Math 21b Apr 11: Nonlinear Systems and the operator D

Qualitative Phase Plane Analysis: The steps for phase plane analysis of the nonlinear system $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$ are:

- Draw the $\frac{dx}{dt} = 0$ **nullcline**, indicated by <u>vertical</u> dashes.
- Draw the $\frac{dy}{dt} = 0$ **nullcline**, indicated by <u>horizontal</u> dashes.
- Find the **equilibrium points** by computing <u>the points of intersection of the two nullclines</u>.
- **Orient each nullcline** in each region by testing points. Then orient each region cut out by the nullclines.
- Determine the **stability of each equilibrium point** (a,b) by linearizing using the *Jacobian matrix*

$$A = \begin{bmatrix} \frac{\partial}{\partial x} f(a,b) & \frac{\partial}{\partial y} f(a,b) \\ \frac{\partial}{\partial x} g(a,b) & \frac{\partial}{\partial y} g(a,b) \end{bmatrix}$$

1. Consider the following model for a predator-prey relationship. $x(t)$ represents the population (in hundreds) of the predator species X at time t, and $y(t)$ represents the population (in hundreds) of the prey species Y at time t.

$$\begin{cases} \dfrac{dx}{dt} = x(-4+y) \\ \dfrac{dy}{dt} = y(10-2x-y) \end{cases}$$

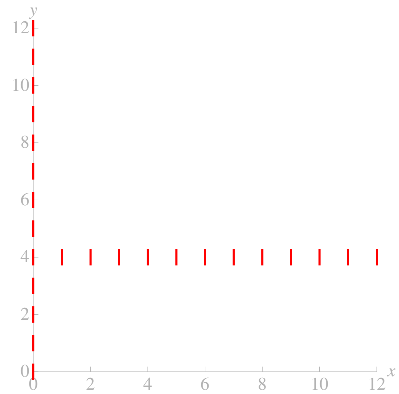
(a) Perform a qualitative phase plane analysis.

Solution: This comes straight from Janet Chen's worksheets. When we do a qualitative phase plane analysis, we are basically trying to sketch the equilibrium points as well as all horizontal and vertical arrows in the direction field. We also want to get the general direction of arrows in the direction field (by "general direction," we mean: up and right vs. up and left vs. down and right vs. down and left). We'll do this in a series of steps.

- **Step 1: Figure out when $\frac{dx}{dt} = 0$.**

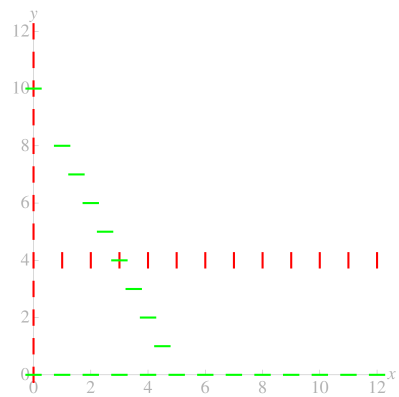
  When $\frac{dx}{dt} = 0$, the arrows in the direction field are vertical. We'll wait to decide whether these arrows go up or down; for now, we'll just draw short vertical line segments at these places.

  From the given equation $\frac{dx}{dt} = x(-4+y)$, we see that $\frac{dx}{dt} = 0$ when $x = 0$ or $y = 4$.



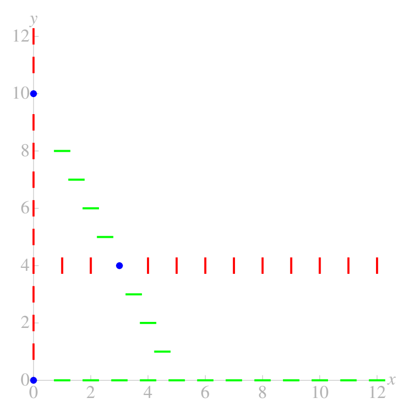
- **Step 2: Figure out when $\frac{dy}{dt} = 0$.**

  When $\frac{dy}{dt} = 0$, the arrows in the direction field are horizontal. Here, $\frac{dy}{dt} = 0$ when $y = 0$ or $10 - 2x - y = 0$ (the latter equation can be rewritten as $y = 10 - 2x$). Along each of these nullclines, we draw short horizontal line segments:



- **Step 3: Identify the equilibrium points, and draw dots there.**

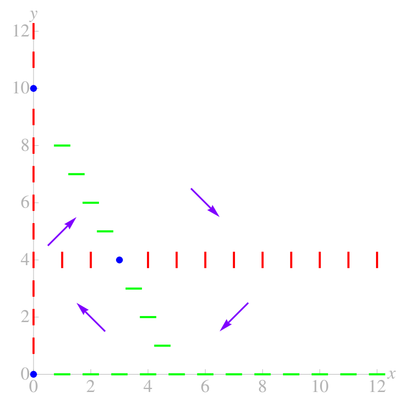
  The equilibrium points are where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. In other words, these are where the red vertical lines and green horizontal lines cross. You can find these just by looking at the sketch so far (they happen at $(0,0)$, $(10,0)$, and $(3,4)$). We will draw the equilibrium points using blue dots (replacing the red and green segments that were there):



- **Step 4: Orient each region.**

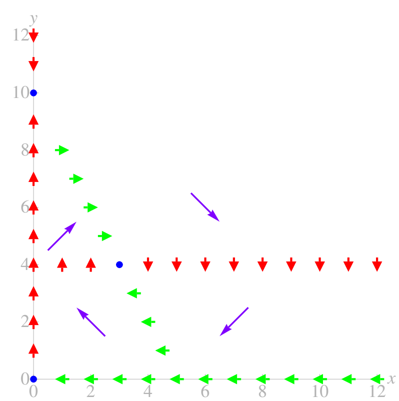
  Notice that the nullclines have divided the first quadrant of our plane into 4 regions. Within a single region, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ cannot change sign (for instance, if $\frac{dx}{dt}$ were to change from positive to negative, it would have to be 0, but this can only happen along a $\frac{dx}{dt} = 0$ nullcline). So, test one point in each region to decide the general direction of trajectories in that region (by "general direction," we mean "up and left," "up and right," "down and left," or "down and right").

  For the top right region, we can pick a point like $(6, 6)$ (any point is fine, as long as you don't pick one along a nullcline). If we plug $x = 6, y = 6$ into the given equations, we get $\frac{dx}{dt} = 12$ and $\frac{dy}{dt} = 6(-8)$. So, $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} < 0$, which means that arrows in the direction field here go right and down. If you do this for the other 3 regions, you get the following diagram:



- **Step 5: Orient the nullclines**

  Now, we can orient the nullclines simply by making them "agree" with the regions.



(b) Which equilibrium points are stable? Use this to draw trajectories.

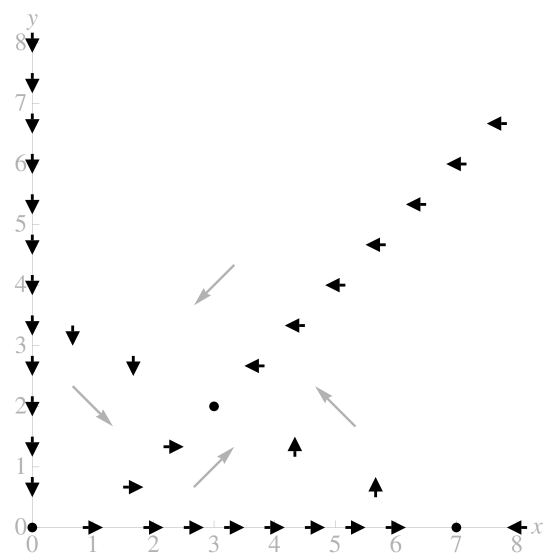
Solution: In some cases, we can see this directly from the phase plane analysis. For example, around $(0, 0)$, all trajectories other than those on the x-axis tend away from $(0, 0)$. Around $(0, 10)$, trajectories along the y-axis tend to $(0, 10)$, but all other trajectories tend away. So,

$\boxed{(0,0) \text{ and } (0,10) \text{ are not asymptotically stable}}$.

To determine whether $(3,4)$ is asymptotically stable, we'll linearize there. The Jacobian of our system is $J(x,y) = \begin{bmatrix} -4+y & x \\ -2y & 10-2x-2y \end{bmatrix}$. So, $J(3,4) = \begin{bmatrix} 0 & 3 \\ -8 & -4 \end{bmatrix}$. This has eigenvalues $-2 \pm 2i\sqrt{5}$, so the trajectories of the linearized system are inward spirals. Since the eigenvalues have negative real part, $\boxed{(3,4) \text{ is asymptotically stable}}$.

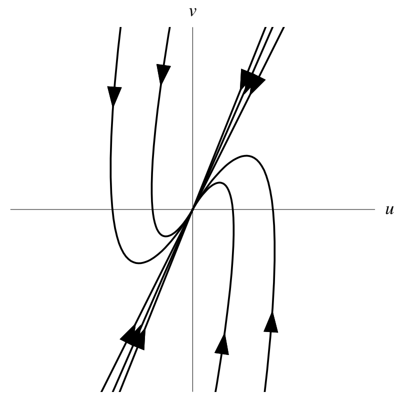
2. Perform a phase plane analysis of the system $\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{3}x(7-x-2y) \\ \dfrac{dy}{dt} = 5y(-1+x-y) \end{cases}$. Use this analysis to draw some trajectories.

Solution: This solution also comes from Janet's worksheets. Here is the phase plane analysis.

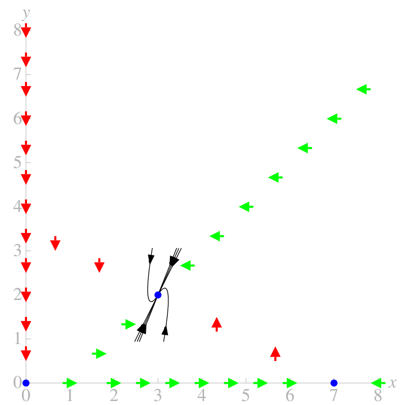


The Jacobian of this system at any point is $J(x,y) = \begin{bmatrix} \frac{1}{3}(7-2x-2y) & \frac{1}{3}(-2x) \\ 5y & 5(-1+x-2y) \end{bmatrix}$.

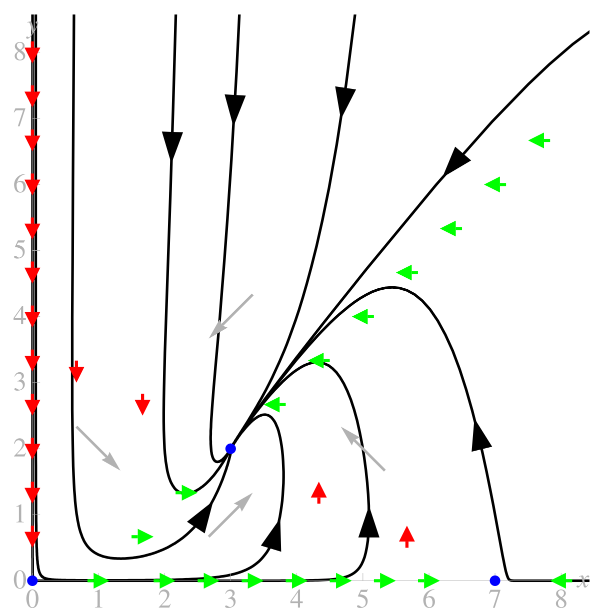
At the equilibrium point $(3,2)$, the Jacobian is $J(3,2) = \begin{bmatrix} -1 & -2 \\ 10 & -10 \end{bmatrix}$. Thus, near $(3,2)$, the trajectories will look like the trajectories of $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} -1 & -2 \\ 10 & -10 \end{bmatrix} \vec{x}$. The matrix $\begin{bmatrix} -1 & -2 \\ 10 & -10 \end{bmatrix}$ has eigenvalues -6 and -5, with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So, the trajectories of $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} -1 & -2 \\ 10 & -10 \end{bmatrix} \vec{x}$ look like this:



So, here are the approximate trajectories near $(3, 2)$:



Here are some actual trajectories of the nonlinear system:



3. In this question, we will consider the operator D defined by $D(f) = f'$ for any smooth function f.

   (a) Remember that we write C^∞ to mean the (infinite-dimensional) linear space of all smooth real-valued functions. Then D is a linear transformation from C^∞ to C^∞ because it satisfies three properties: what are they?

   - $D(0) = 0$, where here 0 refers to the zero function.
   - $D(f+g) = D(f) + D(g)$.
   - $D(c \cdot f) = c \cdot D(f)$ for any constant $c \in \mathbb{R}$.

   All three of these properties are clear.

   (b) Can you describe the *kernel* of D - i.e. the functions f such that $D(f) = 0$?

   Solution: The kernel consists of the *constant functions*. Therefore, it is one-dimensional, as it is spanned by the constant function 1.

   (c) Let λ be a real number. When is λ an *eigenvalue* of D? Can you write down an equation to find an associated *eigenfunction*?

   Solution: The differential equation $\frac{df}{dt} = \lambda f$ can be solved by separation of variables to get $f(t) = Ce^{\lambda t}$ for an arbitrary constant C. In other words, *every* real number λ is an eigenvalue of D, with associated eigenfunction $e^{\lambda t}$.

   (d) Let $f_\lambda(t)$ be the eigenfunction you found in part (c). What is $(D^2+5)f_\lambda$?

   Solution: $(D^2+5)f = f''+5f$, so $(D^2+5)e^{\lambda t} = (\lambda^2+5)e^{\lambda t}$. So D^2+5 has the same eigenfunctions as D, but the associated eigenvalues are different.

   (e) Consider the differential equation $f'' - 4f' + 3f = 0$. Write this equation in the form $A(f) = 0$, where A is a linear transformation formed using D.

   Solution: Since $f'' = D^2(f)$, we can rewrite the above differential equation as $(D^2-4D+3)f = 0$. In other words, the solutions to this differential equation are precisely the kernel of the *differential operator* $D^2 - 4D + 3$.

   (f) Use this form to write down the general solution to the differential equation $f'' - 4f' + 3f = 0$.

Solution: $D^2 - 4D + 3 = (D-1)(D-3)$. Since $\ker(D-1)$ and $\ker(D-3)$ are each one-dimensional, $\ker(D-1)(D-3)$ is at most two-dimensional. In fact, since $\ker(D-1)$ and $\ker(D-3)$ are linearly independent (spanned by e^t and e^{3t}, respectively), and both contained in $\ker(D-1)(D-3)$, it follows that $\ker(D-1)(D-3) = \langle e^t, e^{3t} \rangle$. That is, the general solution to the differential equation is $\boxed{C_1e^t + C_2e^{3t}}$.

4. In this question, we will analyze the linear space C^∞_{per} of smooth real-valued functions $f(t)$ which are 2π*-periodic*. That is, $f(t) = f(t + 2\pi)$ for every t.

(a) Can you think of a few familiar functions which are 2π-periodic?

Solution: $\sin(t), \cos(t)$. More generally, $\sin(nt), \cos(nt)$ for an integer n. (Note that we only need to consider $n \geq 0$ because $\cos(-nt) = \cos(nt)$ and $\sin(-nt) = -\sin(nt)$.)

(b) For the functions f you wrote down in part (a), calculate $D^2 f$ - i.e., the second derivative.

Solution: $D^2(\sin(t)) = D(\cos(t)) = -\sin(t)$. Similarly, $D^2(\cos(t)) = -\cos(t)$. Therefore, these functions are eigenfunctions of D^2 with eigenvalue -1. In general, $\cos(nt)$ and $\sin(nt)$ have eigenvalues $-n^2$. (However, they are not eigenfunctions of D!)

(c) What are the *eigenvalues* of D as it acts on the linear space C^∞_{per}?

Solution: Since D^2 has $-n^2$ as an eigenvalue (of multiplicity at least two) for every positive integer n, D has eigenvalues in and $-in$ for every positive integer n. But where are the *eigenfunctions*?

(d) Can you write down the associated *eigenfunctions*? (Note that these will be complex-valued functions!)

Solution: We use the following trick:

$$D(\cos(t) + i\sin(t)) = (-\sin(t) + i\cos(t)) = i(\cos(t) + i\sin(t))$$

$$D(\cos(t) - i\sin(t)) = (-\sin(t) - i\cos(t)) = -i(\cos(t) - i\sin(t))$$

Since $\cos(t) + i\sin(t) = e^{it}$ and $\cos(t) - i\sin(t) = e^{-it}$, we have complex eigenfunctions e^{it} and e^{-it} of eigenvalues i and $-i$. More generally, we have eigenfunctions e^{int}, e^{-int} of eigenvalues $in, -in$.