# Math 21b Apr 11: Nonlinear Systems and the operator $D$

**Qualitative Phase Plane Analysis:** The steps for phase plane analysis of the nonlinear system $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$ are:

- Draw the $\frac{dx}{dt} = 0$ **nullcline**, indicated by ______________ dashes.
- Draw the $\frac{dy}{dt} = 0$ **nullcline**, indicated by ______________ dashes.
- Find the **equilibrium points** by computing ____________________________.
- **Orient each nullcline** in each region by testing points. Then orient each region cut out by the nullclines.
- Determine the **stability of each equilibrium point** $(a,b)$ by linearizing using the *Jacobian matrix*

$$A = \begin{bmatrix} \frac{\partial}{\partial x} f(a,b) & \frac{\partial}{\partial y} f(a,b) \\ \frac{\partial}{\partial x} g(a,b) & \frac{\partial}{\partial y} g(a,b) \end{bmatrix}$$

1. Consider the following model for a predator-prey relationship. $x(t)$ represents the population (in hundreds) of the predator species X at time $t$, and $y(t)$ represents the population (in hundreds) of the prey species Y at time $t$.

$$\begin{cases} \dfrac{dx}{dt} = x(-4+y) \\ \dfrac{dy}{dt} = y(10-2x-y) \end{cases}$$

   (a) Perform a qualitative phase plane analysis.

(b) Which equilibrium points are stable? Use this to draw trajectories.

2. Perform a phase plane analysis of the system $\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{3}x(7 - x - 2y) \\ \dfrac{dy}{dt} = 5y(-1 + x - y) \end{cases}$. Use this analysis to draw some trajectories.

3. In this question, we will consider the operator $D$ defined by $D(f) = f'$ for any smooth function $f$.

(a) Remember that we write $C^\infty$ to mean the (infinite-dimensional) linear space of all smooth real-valued functions. Then $D$ is a linear transformation from $C^\infty$ to $C^\infty$ because it satisfies three properties: what are they?

(b) Can you describe the *kernel* of $D$ - i.e. the functions $f$ such that $D(f) = 0$?

(c) Let $\lambda$ be a real number. When is $\lambda$ an *eigenvalue* of $D$? Can you write down an equation to find an associated *eigenfunction*?

(d) Let $f_\lambda(t)$ be the eigenfunction you found in part (c). What is $(D^2 + 5)f_\lambda$?

(e) Consider the differential equation $f'' - 4f' + 3f = 0$. Write this equation in the form $A(f) = 0$, where $A$ is a linear transformation formed using $D$.

(f) Use this form to write down the general solution to the differential equation $f'' - 4f' + 3f = 0$.

4. In this question, we will analyze the linear space $C^{\infty}_{\text{per}}$ of smooth real-valued functions $f(t)$ which are $2\pi$-*periodic*. That is, $f(t) = f(t + 2\pi)$ for every $t$.

(a) Can you think of a few familiar functions which are $2\pi$-periodic?

(b) For the functions $f$ you wrote down in part (a), calculate $D^2 f$ - i.e., the second derivative.

(c) What are the *eigenvalues* of $D$ as it acts on the linear space $C^{\infty}_{\text{per}}$?

(d) Can you write down the associated *eigenfunctions*? (Note that these will be complex-valued functions!)