MATH 21B, JANUARY 31: MATRICES - ROWS, RANK, AND REDUCED ROW ECHELON FORM

A matrix is said to be in **reduced row echelon form** if it satisfies the following properties:

(1) If a row contains nonzero entries, then the first nonzero entry is a 1, and is called a **leading 1**.
(2) If a column contains a leading 1, then the other entries in that column are 0.
(3) If a row has a leading 1, then every row above it has a leading 1 somewhere to the left.

The number of leading 1's is called the **rank**. Pictorially, a matrix in reduced row echelon form looks something like the following.

$$\begin{bmatrix} 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

where the *'s can be any numbers, and the 1's shown are leading 1's.

(1) In the following systems, use Gauss-Jordan elimination (row operations) to reduce the coefficient matrix to reduced row echelon form. Here, $\vec{x}$ is a *column vector* whose size is equal to the number of variables of the system. How can we then use this form to find all solutions? (Bonus: Can you see a relation between the *rank* of the system and the structure of the solutions?)

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & -4 & -2 \\ 2 & -5 & -4 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 2 & -2 \\ 1 & 0 & 3 & 0 & 4 \\ -1 & 3 & 3 & 0 & -10 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & -4 & -2 \\ 2 & -5 & -4 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

(2) Each of the following matrices is the reduced row echelon form of the augmented matrix of an unknown system. How many solutions does the system have? Explain briefly.

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(3) If you perform Gauss-Jordan elimination on a inconsistent system, how do you recognize that the system is inconsistent?

(4) If A is an $n \times m$ matrix such that $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$, what can you say about $\mathrm{rref}(A)$?

(5) If the reduced row echelon form of a matrix A has a row of all zeroes, what does this imply about the rows of A? If there are two rows of all zeroes?

(6) Find all values of a for which the system $\begin{bmatrix} 2 & a \\ 3 & 6 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is consistent.