MATH 21B, JANUARY 26: INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS

(1) Solve the following systems using the method of elimination:

(a)

$$\left|\begin{array}{rcrcrcr} & - & y & + & 2z & = & 2 \\ x & + & 4y & - & 2z & = & 5 \\ 3x & + & 2y & + & 5z & = & 7 \end{array}\right|$$

Solution: We use the second equation to eliminate occurrences of x from all other equations. Subtract thrice the second equation from the third, and switch that equation to the top of the list.

$$\left|\begin{array}{rcrcrcr} & - & y & + & 2z & = & 2 \\ x & + & 4y & - & 2z & = & 5 \\ 3x & + & 2y & + & 5z & = & 7 \end{array}\right| \to \left|\begin{array}{rcrcrcr} & - & y & + & 2z & = & 2 \\ x & + & 4y & - & 2z & = & 5 \\ & - & 10y & + & 11z & = & -8 \end{array}\right| \to \left|\begin{array}{rcrcrcr} x & + & 4y & - & 2z & = & 5 \\ & - & y & + & 2z & = & 2 \\ & - & 10y & + & 11z & = & -8 \end{array}\right|$$

Now, use the second equation to cancel out all occurrences of y from the other equations.

$$\left|\begin{array}{rcrcrcr} x & + & 4y & - & 2z & = & 5 \\ & - & y & + & 2z & = & 2 \\ & - & 10y & + & 11z & = & -8 \end{array}\right| \to \left|\begin{array}{rcrcrcr} x & & & + & 6z & = & 13 \\ & - & y & + & 2z & = & 2 \\ & & & - & 9z & = & -28 \end{array}\right| \to \left|\begin{array}{rcrcrcr} x & & & + & 6z & = & 13 \\ & & y & - & 2z & = & -2 \\ & & & - & 9z & = & -28 \end{array}\right|$$

Finally, use the last equation to cancel out all occurrences of z from earlier equations.

$$\left|\begin{array}{rcrcrcr} x & & & + & 6z & = & 13 \\ & & y & - & 2z & = & -2 \\ & & & - & 9z & = & -28 \end{array}\right| \to \left|\begin{array}{rcrcrcr} x & & & + & 6z & = & \frac{117}{9} \\ & & y & - & 2z & = & -\frac{18}{9} \\ & & & & z & = & \frac{28}{9} \end{array}\right| \to \left|\begin{array}{rcrcrcr} x & & & & & = & -\frac{51}{9} \\ & & y & & & = & \frac{38}{9} \\ & & & & z & = & \frac{28}{9} \end{array}\right|$$

Thus, the solution is $x = -\frac{17}{3}, y = \frac{38}{9}, z = \frac{28}{9}$.

(b)

$$\left|\begin{array}{rcrcrcr} 2x & + & 4y & - & 2z & = & -10 \\ 3x & + & 6y & & & = & -12 \end{array}\right|$$

Solution: Scale both rows, and then use the second row to eliminate all occurrences of x.

$$\left|\begin{array}{rcrcrcr} 2x & + & 4y & - & 2z & = & -10 \\ 3x & + & 6y & & & = & -12 \end{array}\right| \to \left|\begin{array}{rcrcrcr} x & + & 2y & - & z & = & -5 \\ x & + & 2y & & & = & -4 \end{array}\right| \to \left|\begin{array}{rcrcrcr} & & & - & z & = & -1 \\ x & + & 2y & & & = & -4 \end{array}\right|$$

Thus, $z = 1$, and $x + 2y = -4$. y is therefore a *free* variable, while x, z are the *leading* variables. The general solution to the equation is $x = -4 - 2t, y = t, z = 1$, where t can be any real number.

(c)

$$\left|\begin{array}{rcrcr} x & + & y & = & 2 \\ 2x & - & y & = & 1 \\ -x & + & y & = & 1 \end{array}\right|$$

Solution: Use the first equation to eliminate occurrences of x, and then use the second equation to eliminate occurrences of y.

$$\left|\begin{array}{rcrcr} x & + & y & = & 2 \\ 2x & - & y & = & 1 \\ -x & + & y & = & 1 \end{array}\right| \to \left|\begin{array}{rcrcr} x & + & y & = & 2 \\ & - & 5y & = & -3 \\ & + & 2y & = & 3 \end{array}\right| \to \left|\begin{array}{rcrcr} x & + & y & = & 2 \\ & & y & = & -\frac{5}{3} \\ & + & 2y & = & 3 \end{array}\right|$$

$$\to \left|\begin{array}{rcrcr} x & + & y & = & 2 \\ & & y & = & -\frac{5}{3} \\ & & & = & \frac{19}{3} \end{array}\right|$$

In other words, if these three equations had a solution (x, y), then we could use this solution to show that $0 = \frac{19}{3}$. But this equation is clearly impossible! Therefore, there is no solution.

(d)

$$\left|\begin{array}{rcrcr} x & + & 2y & = & 3 \\ x & + & 2y & = & 7 \end{array}\right|$$

Solution: Row reducing as in the previous problem gives $0 = 4$, so again, there are no solutions.

(2) For which values of k is the following system inconsistent? One solution? More than one solution?

$$\left|\begin{array}{rcrcrcr} 2x & + & 2y & + & kz & = & 3 \\ kx & + & ky & + & 8z & = & k+2 \end{array}\right|$$

Solution: Use elimination as usual.

$$\left|\begin{array}{rcrcrcr} 2x & + & 2y & + & kz & = & 3 \\ kx & + & ky & + & 8z & = & k+2 \end{array}\right| \to \left|\begin{array}{rcrcrcr} x & + & y & + & \frac{k}{2}z & = & \frac{3}{2} \\ kx & + & ky & + & 8z & = & k+2 \end{array}\right|$$

$$\to \left|\begin{array}{rcrcrcr} x & + & y & + & \frac{k}{2}z & = & \frac{3}{2} \\ & & & & \left(8 - \frac{k^2}{2}\right)z & = & 2 - \frac{k}{2} \end{array}\right|$$

Now, the next step we take depends on whether $8 - \frac{k^2}{2} = 0$

- If $8 - \frac{k^2}{2} \neq 0$, then we can continue our elimination by dividing the second row by $8 - \frac{k^2}{2}$.

$$\to \left|\begin{array}{rcrcrcr} x & + & y & + & \frac{k}{2}z & = & \frac{3}{2} \\ & & & & z & = & \left(2 - \frac{k}{2}\right) / \left(8 - \frac{k^2}{2}\right) \end{array}\right|$$

Even though we're not done with the elimination, we can see that the system will be consistent after we use the second equation to cancel out the z's from the first equation. We will have one free variable (namely, y), and thus infinitely many solutions.

- If $8 - \frac{k^2}{2} = 0$ (which happens when $k^2 = 16$, or $k = \pm 4$), our array becomes

$$\left| \begin{array}{ccccccc} x & + & y & + & \frac{k}{2}z & = & \frac{3}{2} \\ & & & & & = & 2 - \frac{k}{2} \end{array} \right|$$

If $2 - \frac{k}{2} \neq 0$ (that is, $k neq 4$), then this system is inconsistent, and have no solutions. (Since we are currently considering the case where $k = \pm 4$, we are really just talking about $k = -4$ right now.)

If $2 - \frac{k}{2} = 0$ (that is, if $k = 4$), then this system has infinitely many solutions (because it has two free variables).

Putting everything together, we find that if $k = -4$, the system has no solutions, and otherwise, it has infinitely many solutions.

Note: There is a geometric interpretation. The two equations represent planes in $\mathbb{R}^3$. If the two planes are parallel (but not equal), then the system has no solutions; otherwise, the system has infinitely many solutions (do you see why?).