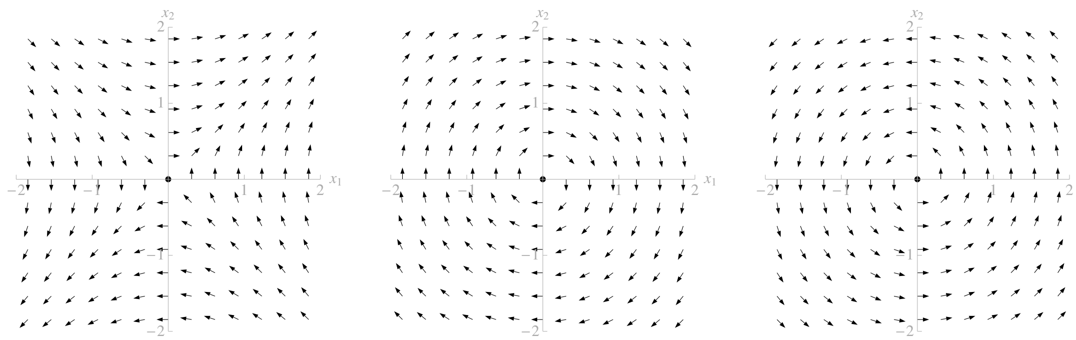
# Math 21b Apr 6: Linear Continuous Dynamical Systems

1. (Warmup) Let $k$ be a constant. What is the solution to the differential equation $\frac{dx}{dt} = kx$ with a given initial condition $x(0)$?

   **Solution:** $\boxed{x(t) = e^{kt}x(0)}$. You can explicitly check that any function $Ce^{kt}$, where $C$ is a constant, satisfies the differential equation - i.e., its derivative is $k$ times itself. Alternatively, the differential equation can be solved directly by separation of variables.

2. (a) Which of the following is the direction field of the continuous linear dynamical system $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x}$?



(A) (B) (C)

   **Solution:** It's (B). You can check this, for example, by computing $\frac{d\vec{x}}{dt}$ for some values of $\vec{x}$, such as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

   (b) Based on the direction field, can you guess the solution of $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x}$ with $\vec{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$? Check that your guess is really a solution.

   **Solution:** The solution with $\vec{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is $\boxed{\vec{x}(t) = \begin{bmatrix} 2\cos(t) \\ 2\sin(t) \end{bmatrix}}$. We can directly confirm that this satisfies the differential equation.

3. Suppose we have a continuous dynamical system $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A$ is an $n \times n$ matrix. Moreover, suppose that $A$ has eigenbasis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ with eigenvectors $\lambda_1, \ldots, \lambda_n$. Thus, let $A = SDS^{-1}$ where $D$ is a diagonal matrix whose diagonal entries are the eigenvalues, and $S$ is the matrix whose columns are the eigenvectors. In this problem, we will use this information to explicitly solve for $\vec{x}(t)$, given an initial condition $\vec{x}(0)$.

   (a) How can we express the vector $\vec{x}(t)$ in terms of the eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$?

**Solution:** By assumption, $\vec{v}_1, \ldots, \vec{v}_n$ form a *basis* for $\mathbb{R}^n$. Therefore, $\vec{x}(t)$ can be expressed as a linear combination

$$\vec{x}(t) = c_1(t)\vec{v}_1 + \ldots + c_n(t)\vec{v}_n$$

where $c_1(t), \ldots, c_n(t)$ are some *scalar*-valued functions. Explicitly, one can compute $c_1(0), \ldots, c_n(0)$ by computing $S^{-1}\vec{x}(0)$, because

$$\vec{x}(t) = S \begin{bmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{bmatrix} \implies S^{-1}\vec{x}(t) = \begin{bmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{bmatrix}$$

(b) Rewrite the differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$ in terms of your answer to (a), and use that to find the general solution of the continuous dynamical system.

**Solution:** Plug the new expression for $\vec{x}(t)$ into both sides of the differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$.

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}(c_1(t)\vec{v}_1 + \ldots + c_n(t)\vec{v}_n) = \frac{dc_1}{dt}\vec{v}_1 + \ldots + \frac{dc_n}{dt}\vec{v}_n$$

$$A\vec{x} = c_1(t)(A\vec{v}_1) + \ldots + c_n(t)(A\vec{v}_n) = c_1(t)\lambda_1\vec{v}_1 + \ldots + c_n(t)\lambda_n\vec{v}_n$$

These two expressions on the far right are equal. Since the vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent, this tells us that

$$\frac{dc_k}{dt} = \lambda_k c_k(t) \qquad k = 1, 2, \ldots, n$$

and therefore

$$c_k(t) = c_k(0)e^{\lambda_k t} \qquad k = 1, 2, \ldots, n$$

Therefore, if we write $a_1 = c_1(0), \ldots, a_n = c_n(0)$, we have

$$\boxed{\vec{x}(0) = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n \implies \vec{x}(t) = a_1 e^{\lambda_1 t}\vec{v}_1 + \ldots + a_n e^{\lambda_n t}\vec{v}_n}$$

(c) What is the solution $\vec{x}(t)$ of $\frac{d\vec{x}}{dt} = A\vec{x}$ satisfying the initial condition $\vec{x}(0) = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$?

**Solution:** Ah, I guess I answered this above.

4. Consider the continuous dynamical system $\frac{d\vec{x}}{dt} = \begin{bmatrix} -2 & 0 \\ -6 & 1 \end{bmatrix}\vec{x}$. We are given that $\begin{bmatrix} -2 & 0 \\ -6 & 1 \end{bmatrix}$ has eigenvalues $-2, 1$ with corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) Find the solution satisfying the initial condition $\vec{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

**Solution:** We first express $\vec{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as a linear combination of the two eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by computing

$$S^{-1}\vec{x}(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

thus $\vec{x}(0) = 2\begin{bmatrix}1\\2\end{bmatrix} - 3\begin{bmatrix}0\\1\end{bmatrix}$. Thus, $\boxed{\vec{x}(t) = 2e^{-2t}\begin{bmatrix}1\\2\end{bmatrix} - 3e^{t}\begin{bmatrix}0\\1\end{bmatrix}}$.

(b) Sketch the full phase portrait of the continuous dynamical system.

I can't type a picture so easily, but the phase portrait will look analogous to 6(f).

(c) Is this system stable?

**Solution:** No, it is not. In fact, the only solutions $\vec{x}(t)$ which converge to zero are those with $\vec{x}(0) = a_1\begin{bmatrix}1\\2\end{bmatrix}$.

**Stability:** We say that a linear continuous dynamical system is *asymptotically stable* (or just *stable*) if all trajectories go to $\vec{0}$ as $t \to \infty$.

5. Let $A = \begin{bmatrix}-4 & 6\\-3 & 2\end{bmatrix}$, and consider the continuous dynamical system $\frac{d\vec{x}}{dt} = A\vec{x}$. The eigenvalues of $A$ are $-1 \pm 3i$.

(a) Describe the trajectories of the system.

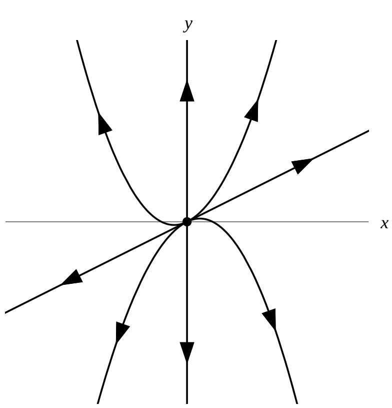
**Solution:** The trajectories are inward spirals converging to 0. This is because $e^{(-1+3i)t} = e^{-t}(\cos(t) + i\sin(t))$ and $e^{(-1-3i)t} = e^{-t}(\cos(t) - i\sin(t))$: so as $t \to \infty$, the magnitude $e^{-t}$ goes to zero.

(b) Is this system stable? Generalize: if $A$ is any $n \times n$ matrix which is diagonalizable over $\mathbb{C}$, how can we tell from the eigenvalues of $A$ whether the system is stable?

**Solution:** Yes, it is stable. In general, $A$ is stable if and only if as $t \to \infty$, $e^{\lambda t} \to 0$ for every eigenvalue $\lambda$. This occurs if and only if $|e^{\lambda}| < 1$, i.e. if $\lambda$ has **real part less than 0.**

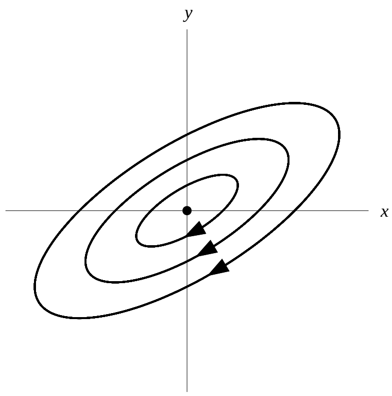
6. Each of the following is the phase portrait of a continuous dynamical system $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A$ is a real $2 \times 2$ matrix. What can you say about the eigenvalues of $A$? In which cases is the system stable?

(a)



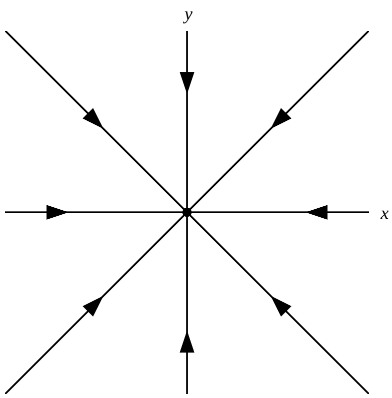

**Solution:** The eigenvalues are both positive (because all trajectories go outward) and distinct (because there are curved trajectories). The system is not stable.

(b)



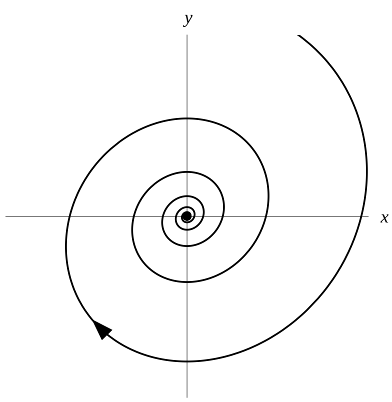

**Solution:** The eigenvalues are complex conjugates with real part zero (because the trajectories are closed loops), i.e. they are **purely imaginary**, because $e^{\lambda t}$ is periodic only when $\lambda$ is purely imaginary. The system is not stable.

(c)



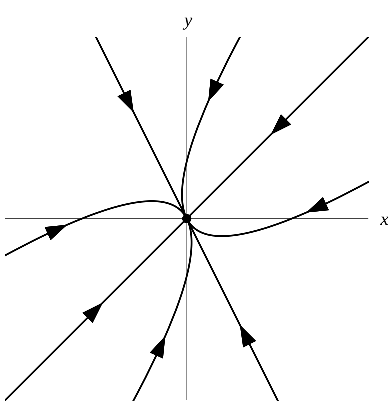

**Solution:** The eigenvalues are both negative (because all trajectories go inward) and equal (because the trajectories are all straight lines). The system is stable.

(d)



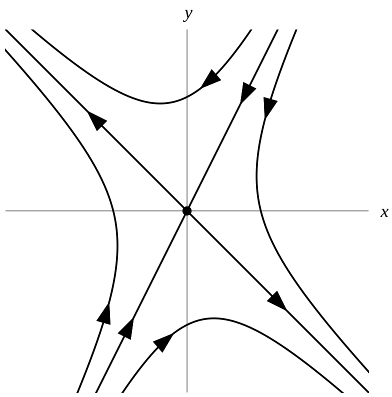

**Solution:** The eigenvalues are complex conjugates with negative real part (because the trajectories spiral inwards). The system is stable.

(e)




**Solution:** The eigenvalues are both negative and are distinct. The system is stable.

(f)




**Solution:** One eigenvalue is positive, and one is negative. The system is not stable.