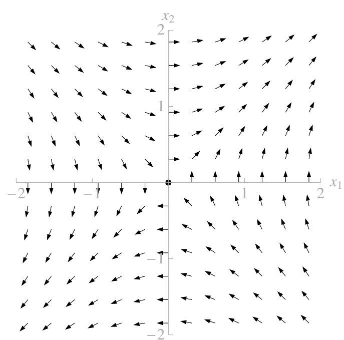
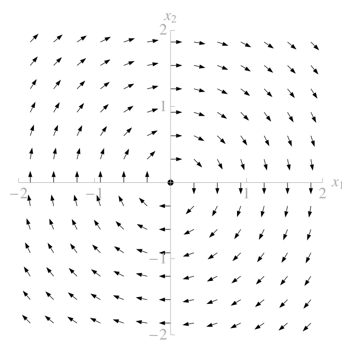
# Math 21b Apr 6: Linear Continuous Dynamical Systems

1. (Warmup) Let $k$ be a constant. What is the solution to the differential equation $\frac{dx}{dt} = kx$ with a given initial condition $x(0)$?

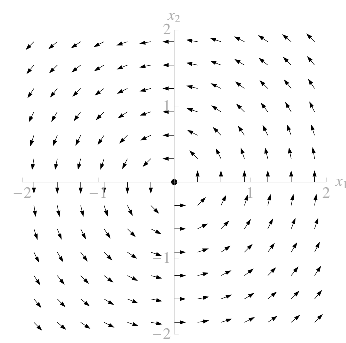
2. (a) Which of the following is the direction field of the continuous linear dynamical system $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$?



(A)



(B)



(C)

   (b) Based on the direction field, can you guess the solution of $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$ with $\vec{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$? Check that your guess is really a solution.

3. Suppose we have a continuous dynamical system $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A$ is an $n \times n$ matrix. Moreover, suppose that $A$ has eigenbasis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ with eigenvectors $\lambda_1, \ldots, \lambda_n$. In this problem, we will use this information to explicitly solve for $\vec{x}(t)$, given an initial condition $\vec{x}(0)$.

   (a) How can we express the vector $\vec{x}(t)$ in terms of the eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$?

(b) Rewrite the differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$ in terms of your answer to (a), and use that to find the general solution of the continuous dynamical system.

(c) What is the solution $\vec{x}(t)$ of $\frac{d\vec{x}}{dt} = A\vec{x}$ satisfying the initial condition $\vec{x}(0) = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$?

4. Consider the continuous dynamical system $\frac{d\vec{x}}{dt} = \begin{bmatrix} -2 & 0 \\ -6 & 1 \end{bmatrix} \vec{x}$. We are given that $\begin{bmatrix} -2 & 0 \\ -6 & 1 \end{bmatrix}$ has eigenvalues $-2, 1$ with corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) Find the solution satisfying the initial condition $\vec{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(b) Sketch the full phase portrait of the continuous dynamical system.

(c) Is this system stable?

**Stability:** We say that a linear continuous dynamical system is *asymptotically stable* (or just *stable*) if all trajectories go to $\vec{0}$ as $t \to \infty$.

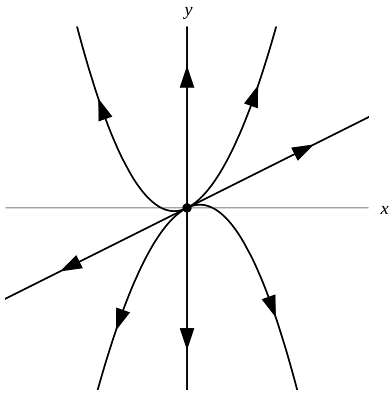
5. Let $A = \begin{bmatrix} -4 & 6 \\ -3 & 2 \end{bmatrix}$, and consider the continuous dynamical system $\frac{d\vec{x}}{dt} = A\vec{x}$. The eigenvalues of $A$ are $-1 \pm 3i$.

   (a) Describe the trajectories of the system.

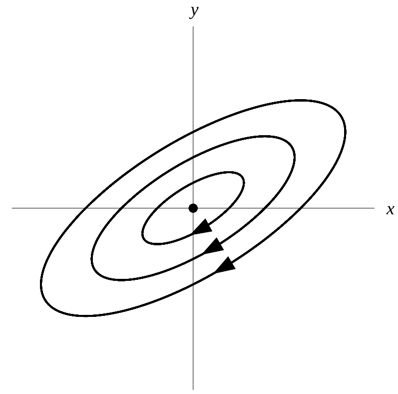
   (b) Is this system stable? Generalize: if $A$ is any $n \times n$ matrix which is diagonalizable over $\mathbb{C}$, how can we tell from the eigenvalues of $A$ whether the system is stable?

6. Each of the following is the phase portrait of a continuous dynamical system $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A$ is a real $2 \times 2$ matrix. What can you say about the eigenvalues of $A$? In which cases is the system stable?

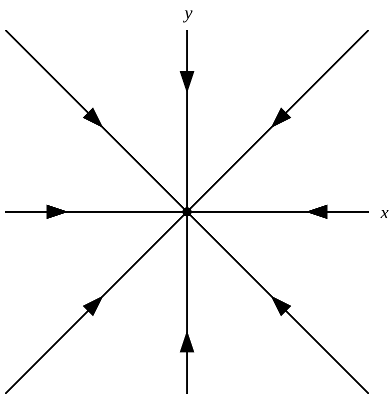
(a)



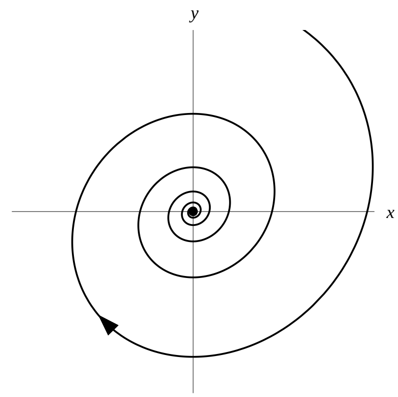

(b)



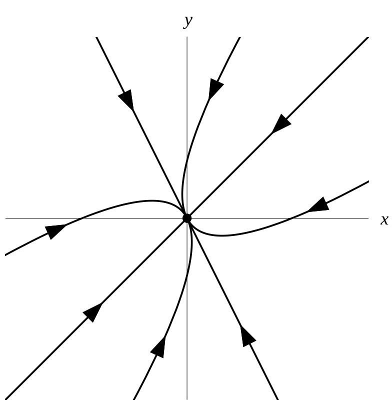

(c)




(d)




(e)




(f)