Math 21b, March 30: Stability (cont.), Symmetric Matrices and the Spectral Theorem

1. For each of the following matrices A, determine whether it is stable (i.e., asymptotically stable).

   (a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

   Not stable: $A^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, thus the initial condition $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gives $\vec{x}(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$ which goes off to infinity.

   (b) $A = \begin{bmatrix} 0.99 & 1000 \\ 0 & 0.9 \end{bmatrix}$

   Stable: Has distinct eigenvalues 0.99 and 0.9, and therefore it's diagonalizable with eigenvalues of magnitude less than 1 $\implies$ stable.

2. Which rotation dilation matrices have trajectories which are circles?

   Solution: In order for the trajectory to be a closed loop, we must have eigenvalues satisfying $|\lambda| = 1$. These are the **rotation matrices** (i.e. no dilation component) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. (Note this matrix, which represents rotation counterclockwise by θ around the origin, has eigenvalues $e^{\pm i\theta}$.)

The **dot product** for **complex vectors** is defined by the formula

$$v \cdot w = \overline{v}^T w$$

($\overline{v}$ is the complex conjugate of the vector v.) If A is any matrix, then $v \cdot (Aw) = (\overline{A}^T v) \cdot w$. This is by algebraic manipulation:

$$v \cdot (Aw) = \overline{v}^T (Aw) = (\overline{v}^T A) w = \overline{\overline{A}^T v}^T w = (\overline{A}^T v) \cdot w$$

A is called **symmetric** if $A^T = A$.

A is called **anti-symmetric** if $A^T = -A$.

3. Suppose that A is a symmetric matrix with real entries. In this problem, we will show that its **eigenvalues are real** and it has an **eigenbasis of orthonormal vectors.**

   (a) Show that $(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A\vec{y})$ for any vectors $\vec{x}, \vec{y}$.

   Solution:
   $$\vec{x} \cdot (A\vec{y}) = (\overline{A}^T \vec{x}) \cdot \vec{y} = (A^T \vec{x}) \cdot \vec{y} = (A\vec{x}) \cdot \vec{y}$$

   The first step is by the property in the first box above. The second step is because A has all real entries, and the third step is because A is symmetric.

(b) Suppose that $\vec{v}$ is a unit vector with eigenvalue λ. Argue that λ must be *real* by comparing $(A\vec{v})\cdot\vec{v}$ and $\vec{v}\cdot(A\vec{v})$.

Solution: By part (a), $(A\vec{v})\cdot\vec{v} = \vec{v}\cdot(A\vec{v})$. Separately calculating each,

$$\vec{v}\cdot(A\vec{v}) = \vec{v}\cdot(\lambda\vec{v}) = \lambda(\vec{v}\cdot\vec{v})$$
$$(A\vec{v})\cdot\vec{v} = (\lambda\vec{v})\cdot\vec{v} = \overline{\lambda}(\vec{v}\cdot\vec{v})$$

Thus, $\lambda = \overline{\lambda}$ and so λ is real.

(c) Suppose that $\vec{v}_1$ and $\vec{v}_2$ have *different* eigenvalues λ_1 and λ_2, respectively. Argue that $\vec{v}_1$ and $\vec{v}_2$ are *orthogonal* by comparing $(A\vec{v}_1)\cdot\vec{v}_2$ and $\vec{v}_1\cdot(A\vec{v}_2)$.

Solution:

$$\lambda_1(\vec{v}_1\cdot\vec{v}_2) = (A\vec{v}_1)\cdot\vec{v}_2 = \vec{v}_1\cdot(A\vec{v}_2) = \lambda_2(\vec{v}_1\cdot\vec{v}_2)$$

The first step is because $\vec{v}_1$ is an eigenvector, the middle step is by part (a), and the last step is because $\vec{v}_2$ is an eigenvector. Therefore, $(\lambda_1-\lambda_2)(\vec{v}_1\cdot\vec{v}_2) = 0$. Since $\lambda_1-\lambda_2 \neq 0$, it follows that $\vec{v}_1\cdot\vec{v}_2 = 0$, i.e. $\vec{v}_1 \perp \vec{v}_2$.

(d) Can you argue that if A is symmetric, then it must be diagonalizable? (This is not so easy!)

Solution: Two proofs of this fact will be given at the end of this worksheet.

4. Suppose that A has **real eigenvalues** and has an **orthonormal eigenbasis**. That is, $A = SDS^{-1}$ where D is a real diagonal matrix, and S is *orthogonal.* What is the relationship between A and A^T?

Solution: They are equal, because $S^T = S^{-1}$ and $D^T = D$.

$$A^T = (SDS^{-1})^T = (S^{-1})^T D^T S^T = SDS^{-1} = A$$

The last two problems have proven the **Spectral Theorem.** This theorem states that a real matrix is symmetric if and only if it is diagonalizable with an orthonormal eigenbasis.

5. Let $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$.

(a) Is A diagonalizable over $\mathbb{R}$? If so, find a diagonal matrix that A is similar to.

Solution: A is symmetric, so by the Spectral Theorem, it must be diagonalizable over $\mathbb{R}$. Or, we can prove it by explicitly computing the eigenvalues: they are 5 and -5, and because there are no multiple roots, A must be diagonalizable. Therefore, A is similar to $\begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$ and also similar to $\begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix}$.

(b) Describe the linear transformation $T(\vec{x}) = A\vec{x}$ geometrically.

Solution: It's a **reflection dilation**, because it has an eigenvector of eigenvalue 5, one of eigenvalue -5, and these are orthogonal to each other. The eigenspace of eigenvalue 5 is the line of reflection (which turns out to be the span of $\begin{bmatrix}1\\2\end{bmatrix}$).

(c) Let $M = \begin{bmatrix}997 & 4\\ 4 & 1003\end{bmatrix}$. What is the relationship between the eigenvalues of A and the eigenvalues of M? What about the eigenvectors?

Solution: $M = A + 1000I$. Therefore, M and A have the *same eigenvectors*, and M has eigenvalues 1005 and 995.

6. Let V be the plane $x+2y+3z = 0$ in $\mathbb{R}^3$ and let A be the matrix of reflection over V. Is A symmetric? Explain carefully.

Solution: Yes. Let $\vec{v}_1, \vec{v}_2$ be an orthonormal basis for the plane V, and let $\vec{v}_3$ be a unit vector orthogonal to V. Then $A\vec{v}_1 = \vec{v}_1$, $A\vec{v}_2 = \vec{v}_2$, and $A\vec{v}_3 = -\vec{v}_3$. That is, $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ form an orthonormal eigenbasis for A with eigenvalues $1, 1, -1$. Therefore, by the spectral theorem, A is symmetric.

7. The matrix $A = \begin{bmatrix}101 & 1 & 2\\ 1 & 101 & 2\\ 2 & 2 & 104\end{bmatrix}$ is symmetric. Find its eigenvalues and an *orthogonal* eigenbasis. (no need to normalize the eigenvectors for this question)

Solution: Let's instead work with the matrix $M = \begin{bmatrix}1 & 1 & 2\\ 1 & 1 & 2\\ 2 & 2 & 4\end{bmatrix}$, as M has the same eigenvectors as A, and its eigenvalues are just 100 less. Since $\operatorname{rank}(M) = 2$, M has 0 as an eigenvalue with multiplicity 2. To find eigenvectors, notice that

$$\begin{bmatrix}1 & 1 & 2\\ 1 & 1 & 2\\ 2 & 2 & 4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x+y+2z\\ x+y+2z\\ 2x+2y+4z\end{bmatrix}$$

and the only way that this can be a *nonzero* multiple of $\begin{bmatrix}x\\y\\z\end{bmatrix}$ is if $\begin{bmatrix}x\\y\\z\end{bmatrix}$ is a multiple of $\begin{bmatrix}1\\1\\2\end{bmatrix}$. Plugging this in, we find that $\boxed{\begin{bmatrix}1\\1\\2\end{bmatrix}}$ is an eigenvector with eigenvalue $\boxed{6}$.

The eigenspace of eigenvalue 0 must be the orthogonal complement of the span of $\begin{bmatrix}1\\1\\2\end{bmatrix}$ - i.e., it is the kernel of $\begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$. We can calculate a basis is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and then using Gram-Schmidt to orthogonalize, we get the orthogonal eigenvectors $\boxed{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$ with eigenvalues $\boxed{0}$.

8. True or false: If A and B are similar (i.e. $A = SBS^{-1}$ for some S), then they have the same eigenvalues.

True: They have the same characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det(SBS^{-1} - \lambda I) = \det(SBS^{-1} - \lambda(SIS^{-1})) = \det(S(B - \lambda I)S^{-1}) \\ &= \det(S)\det(B - \lambda I)\det(S^{-1}) = \det(B - \lambda I)\det(S)\det(S^{-1}) = \det(B - \lambda I) \end{aligned}$$

9. True or false: If A and B are matrices with the same eigenvalues and multiplicities, then they are similar.

False: Here is a counterexample. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

10. True or false: If A and B are diagonalizable matrices with the same eigenvalues and multiplicities, then they are similar.

True: If A and B have the same eigenvalues and multiplicities, then $A = SDS^{-1}$ and $B = RDR^{-1}$ where D is the diagonal matrix with the eigenvalues along the diagonal. The second equation can be converted into $R^{-1}BR = D$, and then plugging this into the equation for A,

$$A = SDS^{-1} = S(R^{-1}BR)S^{-1} = (SR^{-1})B(RS^{-1}) = (SR^{-1})B(SR^{-1})^{-1}$$

Thus $A \sim B$.

Now, as promised in 3(d), I will prove that *any real symmetric matrix A is diagonalizable*, which is the missing piece of the proof of the Spectral Theorem. I will provide two proofs.

Proof 1 (Pure algebra): First, I will prove that *A has some nonzero real eigenvector $\vec{x}_1$*. We have shown that the characteristic polynomial has all of its roots real (this follows from 3(b)). Let λ_1 be one of its roots. This means that $\det(A - \lambda_1 I_n) = 0$, and so $\ker(A - \lambda_1 I_n)$ contains some nonzero vector $\vec{x}_1$. By definition, $\vec{x}_1$ is a nonzero real eigenvectors (with eigenvalue λ_1), so this proves my first claim.

Next, let $W_1 \subset \mathbb{R}^n$ be the $(n-1)$-dimensional space orthogonal to $\vec{x}_1$. I claim that A *preserves* W_1: i.e., it sends any vector in W_1 to another vector in W_1. This is true because if $\vec{w} \in W_1$, then $\vec{w} \cdot \vec{x}_1 = 0$, and thus

$$(A\vec{w}) \cdot \vec{x}_1 = \vec{w} \cdot (A\vec{x}_1) = \lambda_1(\vec{w} \cdot \vec{x}_1) = 0$$

and therefore $A\vec{w}$ is in W_1. This proves my second claim.

Now, the same argument can be used to show that W_1 contains an eigenvector $\vec{x}_2$ of eigenvalue λ_2, and then we can let W_2 be the orthogonal complement of $\vec{x}_2$ in W_1, and A preserves W_2. Then we can show W_2 contains an eigenvector $\vec{x}_3$, etc. In this way, we can produce an orthonormal eigenbasis $\vec{x}_1, \ldots, \vec{x}_n$.

Proof 2 (Continuity): Pick some symmetric matrix M such that the characteristic polynomial of $A+hM$ has all roots distinct for all $h \in (0, \epsilon)$, for some small $\epsilon > 0$.[1] Then for each $h \in (0, \epsilon)$, $A + hM$ is diagonalizable (because the geometric multiplicity and algebraic multiplicity of each eigenvalue has to be 1). By 3(c), the (one-dimensional) eigenspaces of $A+hM$ are orthogonal to each other, and thus, $A + hM$ is orthogonally diagonalizable.

For each $h \in (0, \epsilon)$, let $\lambda_1(h), \ldots, \lambda_n(h)$ be the eigenvalues of $A+hM$, chosen so that $\lambda_1(h), \ldots, \lambda_n(h)$ are continuous functions in h. Let $\vec{x}_1(h), \ldots, \vec{x}_n(h)$ be corresponding unit eigenvectors, so that $\vec{x}_1(h), \ldots, \vec{x}_n(h)$ are continuous vector-valued functions in h. That is, for each $h \in (0, \epsilon)$,

$$(A + hM)\vec{x}_i(h) = \lambda_i(h)\vec{x}_i(h) \qquad (1 \le i \le n)$$

Note that because $A + hM$ is symmetric, the vectors $\vec{x}_1(h), \ldots, \vec{x}_n(h)$ are pairwise orthogonal for any fixed h.

Now let $\vec{x}_i = \lim_{h\to 0^+} \vec{x}_i(h)$ and let $\lambda_i = \lim_{h \to 0^+} \lambda_i(h)$. Then

$$A\vec{x}_i = \lim_{h\to 0^+} (A + hM)\vec{x}_i(h) = \lim_{h\to 0^+} \lambda_i(h)\vec{x}_i(h) = \lambda_i\vec{x}_i$$

and so $\vec{x}_1, \ldots, \vec{x}_n$ are eigenvectors of A with eigenvalues $\lambda_1, \ldots, \lambda_n$. Moreover, they are orthonormal, because

$$\vec{x}_i \cdot \vec{x}_j = \lim_{h\to 0^+} \vec{x}_i(h) \cdot \vec{x}_j(h) = \lim_{h\to 0^+} 0 = 0 \qquad i \neq j$$
$$\vec{x}_i \cdot \vec{x}_i = \lim_{h\to 0^+} \vec{x}_i(h) \cdot \vec{x}_i(h) = \lim_{h\to 0^+} 1 = 1$$

Thus, we have produced an orthonormal eigenbasis of A. (The reason this argument fails when A is not symmetric, is because the eigenvectors $\vec{x}_1(h), \ldots, \vec{x}_n(h)$ can collapse onto each other, i.e. their pairwise dot products can converge to 1.)

[1] Roughly, the reason why such a matrix M exists is because a 'generic' degree n polynomial, i.e. where all of the coefficients are chosen at random, will have distinct roots. Therefore, a generic $n \times n$ matrix will have distinct eigenvalues.