Math 21b, March 30: Stability (cont.), Symmetric Matrices and the Spectral Theorem

1. For each of the following matrices A, determine whether it is stable (i.e., asymptotically stable).

   (a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 0.99 & 1000 \\ 0 & 0.9 \end{bmatrix}$

2. Which rotation dilation matrices have trajectories which are circles?

The **dot product** for **complex vectors** is defined by the formula

$$v \cdot w = \overline{v}^T w$$

($\overline{v}$ is the complex conjugate of the vector v.) If A is any matrix, then $v \cdot (Aw) = (\overline{A}^T v) \cdot w$.

A is called **symmetric** if

A is called **anti-symmetric** if

3. Suppose that A is a symmetric matrix with real entries. In this problem, we will show that its **eigenvalues are real** and it has an **eigenbasis of orthonormal vectors.**

   (a) Show that $(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A\vec{y})$ for any vectors $\vec{x}, \vec{y}$.

(b) Suppose that $\vec{v}$ is a unit vector with eigenvalue λ. Argue that λ must be *real* by comparing $(A\vec{v})\cdot\vec{v}$ and $\vec{v}\cdot(A\vec{v})$.

(c) Suppose that $\vec{v}_1$ and $\vec{v}_2$ have *different* eigenvalues λ_1 and λ_2, respectively. Argue that $\vec{v}_1$ and $\vec{v}_2$ are *orthogonal* by comparing $(A\vec{v}_1)\cdot\vec{v}_2$ and $\vec{v}_1\cdot(A\vec{v}_2)$.

(d) Can you argue that if A is symmetric, then it must be diagonalizable? (This is not so easy!)

4. Suppose that A has **real eigenvalues** and has an **orthonormal eigenbasis**. That is, $A = SDS^{-1}$ where D is a real diagonal matrix, and S is *orthogonal.* What is the relationship between A and A^T?

The last two problems have proven the **Spectral Theorem.** This theorem states that a real matrix is symmetric if and only if it is diagonalizable with an orthonormal eigenbasis.

5. Let $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$.

   (a) Is A diagonalizable over $\mathbb{R}$? If so, find a diagonal matrix that A is similar to.

   (b) Describe the linear transformation $T(\vec{x}) = A\vec{x}$ geometrically.

   (c) Let $M = \begin{bmatrix} 997 & 4 \\ 4 & 1003 \end{bmatrix}$. What is the relationship between the eigenvalues of A and the eigenvalues of M? What about the eigenvectors?

6. Let V be the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$ and let A be the matrix of reflection over V. Is A symmetric? Explain carefully.

7. The matrix $A = \begin{bmatrix} 101 & 1 & 2 \\ 1 & 101 & 2 \\ 2 & 2 & 104 \end{bmatrix}$ is symmetric. Find its eigenvalues and an orthogonal eigenbasis.

8. True or false: If A and B are similar (i.e. $A = SBS^{-1}$ for some S), then they have the same eigenvalues.

9. True or false: If A and B are matrices with the same eigenvalues, then they are similar.

10. True or false: If A and B are diagonalizable matrices with the same eigenvalues, then they are similar.