Math 21b, March 28: Complex Eigenvalues and Stability

1. Calculate the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Solution: One can easily compute that the characteristic polynomial is $\det(A - \lambda I) = 1 - \lambda^3 = (1-\lambda)(1+\lambda+\lambda^2)$. The zeroes of this polynomial are the *cube roots of unity*, i.e. the numbers 1, $e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. To reduce clutter, let us write $\omega = e^{2\pi i/3}$. Then this means the three eigenvalues are $1, \omega, \omega^2$.

The eigenvectors can be directly computed

$$\ker \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\ker \begin{bmatrix} -\omega & 0 & 1 \\ 1 & -\omega & 0 \\ 0 & 1 & -\omega \end{bmatrix} = \operatorname{span} \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}$$

$$\ker \begin{bmatrix} -\omega^2 & 0 & 1 \\ 1 & -\omega^2 & 0 \\ 0 & 1 & -\omega^2 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$$

Note: Here's an interpretation of why these are the eigenvectors. The matrix $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ takes a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and outputs the vector $\begin{bmatrix} z \\ x \\ y \end{bmatrix}$, i.e. it just shifts the entries down by one (and knocks the last entry up to the top). Now check what this operation does to the three vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}, \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$.

Complex numbers: A complex number is a number of the form $a + bi$, where $i = \sqrt{-1}$. a is the *real part* and bi is the *imaginary part*. Addition of complex numbers is done componentwise, and multiplication is done using the fact that $i^2 = -1$.

$$(a+bi) + (c+di) = (a+c) + (b+d)i \qquad (a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

Euler's formula: Using Taylor series, one can show that $e^{i\theta} = \cos\theta + i\sin\theta$. This provides us a great tool to multiply complex numbers by using *polar coordinates*, i.e. expressing a complex number as $re^{i\theta} = r\cos\theta + ir\sin\theta$, where r is the *length* of the complex number. I.e., we use the formula

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Length of a complex number: The *length* of a complex number $a + bi$ is found by multiplying it by its *conjugate* and taking the square root.

$$\sqrt{(a+bi)(a-bi)} = \sqrt{a^2+b^2}$$

2. In this question, we'll use polar coordinates to calculate $(1+i)^{10}$.

   (a) Write the complex number $1+i$ in polar coordinates $r^{i\theta}$ by finding its length r and its angle θ.

   Solution: $1+i$ has length $\sqrt{1^2+1^2} = \sqrt{2}$, and angle $\pi/4$. Thus, $\boxed{1+i = \sqrt{2}e^{\pi i/4}}$.

   (b) Use this form to calculate $(1+i)^{10}$. Can you find a geometric description of what multiplication by $1+i$ does in the complex plane?

   $$(1+i)^{10} = (\sqrt{2})^{10} e^{10\pi i/4} = 32e^{5\pi i/2} = 32i$$

   because if we rotate counterclockwise by an angle of $5\pi/2$ from the positive real (x) axis (which is the same as rotating by just $\pi/2$), we will be pointing vertically along the positive imaginary (y) axis in the complex plane.

   In general, for any complex number $re^{i\theta}$, $(1+i)re^{i\theta} = (r\sqrt{2})e^{i(\theta+\pi/4)}$, i.e. multiplication by $1+i$ will lengthen the complex number by a factor of $\sqrt{2}$ and rotate it by $\pi/4$.

Complex conjugates: For any complex number $a+bi$, its *conjugate* is the complex number $a-bi$. In polar coordinates, the conjugate of $re^{i\theta}$ is $re^{-i\theta}$. Complex eigenvalues and eigenvectors of matrices come in *conjugate pairs.*

Dot product of complex vectors: If we have two vectors $\vec{v}, \vec{w}$ with complex entries, then to take their *dot product* $v \cdot w$, we have to perform the multiplication $\overline{v}^T w$. I.e., we have to transpose v AND conjugate all of the entries, then multiply the resulting row vector by the column vector w. This is so that *the dot product of a vector with itself equals the squared length.* For example, if $v = \begin{bmatrix} 3+i \\ 2 \end{bmatrix}$, then

$$v \cdot v = \overline{v}^T v = \begin{bmatrix} 3-i & 2 \end{bmatrix} \begin{bmatrix} 3+i \\ 2 \end{bmatrix} = (3-i)(3+i) + (2)(2) = 10 + 4 = 14$$

Matrix multiplication with complex numbers: Multiplying matrices with complex numbers works just the same as multiplying matrices with real numbers. Row reduction to compute matrix inverses, kernel, etc also works the same.

3. Let $A = \begin{bmatrix} -3 & -9 \\ 1 & -3 \end{bmatrix}$. In this problem, we will solve the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with initial condition $\vec{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Remember that we can derive $\vec{x}(t) = A^t\vec{x}(0)$, so we are essentially trying to calculate A^t.

   (a) Calculate the eigenvalues and eigenvectors of A. Use this to diagonalize A. (Your eigenvectors should have complex entries!)

   Solution: First, the eigenvalues. We compute the characteristic polynomial and then find the roots using the quadratic formula.

$$\det(A-\lambda I)=\lambda^2+6\lambda+18 \implies \lambda=-3\pm 3i$$

Let $\lambda_1=-3+3i$ and $\lambda_2=-3-3i$. Eigenvectors are just computed in the same way.

$$\ker(A-\lambda_1 I)=\ker\begin{bmatrix}-3i & -9\\ 1 & -3i\end{bmatrix}=\operatorname{span}\begin{bmatrix}3i\\1\end{bmatrix}$$

$$\ker(A-\lambda_2 I)=\ker\begin{bmatrix}3i & -9\\ 1 & 3i\end{bmatrix}=\operatorname{span}\begin{bmatrix}-3i\\1\end{bmatrix}$$

(Note that the eigenvector for λ_2 is just the conjugate of the eigenvector for λ_1.)

(b) Write down an expression for the matrix A^t in the form SDS^{-1} with D diagonal, and multiply this by $\vec{x}(0)$ to get $\vec{x}(t)$. (Note, the multiplication is quicker if you first multiply S^{-1} by $\vec{x}(0)$, instead of multiplying matrices.)

Solution: We can express $A=SDS^{-1}$ with $D=\begin{bmatrix}-3+3i & 0\\ 0 & -3-3i\end{bmatrix}$ and $S=\begin{bmatrix}3i & -3i\\ 1 & 1\end{bmatrix}$. We need to calculate $(-3+3i)^t$ and $(-3-3i)^t$. $-3+3i$ has length $3\sqrt{2}$ and angle $3\pi/4$, while $-3-3i$ has length $3\sqrt{2}$ and angle $5\pi/4=-3\pi/4$. Thus,

$$(-3+3i)^t=(3\sqrt{2})^t e^{3\pi i t/4}=(3\sqrt{2})^t\left(\cos\frac{3\pi t}{4}+i\sin\frac{3\pi t}{4}\right)$$

$$(-3-3i)^t=(3\sqrt{2})^t e^{-3\pi i t/4}=(3\sqrt{2})^t\left(\cos\frac{3\pi t}{4}-i\sin\frac{3\pi t}{4}\right)$$

One can compute via row reduction that $S^{-1}=\frac{1}{6i}\begin{bmatrix}1 & 3i\\ -1 & 3i\end{bmatrix}$. Thus,

$$\vec{x}(t)=SD^tS^{-1}\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}3i & -3i\\ 1 & 1\end{bmatrix}\begin{bmatrix}(-3+3i)^t & 0\\ 0 & (-3-3i)^t\end{bmatrix}\frac{1}{6i}\begin{bmatrix}1 & 3i\\ -1 & 3i\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}$$

$$=(3\sqrt{2})^t\begin{bmatrix}3i & -3i\\ 1 & 1\end{bmatrix}\begin{bmatrix}\cos\frac{3\pi t}{4}+i\sin\frac{3\pi t}{4} & 0\\ 0 & \cos\frac{3\pi t}{4}-i\sin\frac{3\pi t}{4}\end{bmatrix}\frac{1}{6i}\begin{bmatrix}1 & 3i\\ -1 & 3i\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}$$

Now proceed to multiply this out.

$$=(3\sqrt{2})^t\begin{bmatrix}3i & -3i\\ 1 & 1\end{bmatrix}\begin{bmatrix}\cos\frac{3\pi t}{4}+i\sin\frac{3\pi t}{4} & 0\\ 0 & \cos\frac{3\pi t}{4}-i\sin\frac{3\pi t}{4}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$=(3\sqrt{2})^t\begin{bmatrix}3i & -3i\\ 1 & 1\end{bmatrix}\begin{bmatrix}\cos\frac{3\pi t}{4}+i\sin\frac{3\pi t}{4}\\ \cos\frac{3\pi t}{4}-i\sin\frac{3\pi t}{4}\end{bmatrix}$$

$$=\boxed{(3\sqrt{2})^t\begin{bmatrix}-6\sin\frac{3\pi t}{4}\\ 2\cos\frac{3\pi t}{4}\end{bmatrix}}$$

(c) In words, how does $\vec{x}(t)$ behave as t increases? (Does it grow or shrink? Which quadrant will it be in?)

Solution: As t increases, we can see that the quadrant will vary, because sin and cos oscillate: in fact, it will be a *counterclockwise spiral* because the x-coordinate is $-\sin$ and the y-coordinate is cos. As $t \to \infty$, the factor at the front grows without bound, and so $\vec{x}(t)$ spirals *outwards.*

Stability: The matrix A associated to the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ is called

- **asymptotically stable** (or just *stable*) if $\vec{x}(t) \to 0$ as $t \to \infty$, for *every* initial condition $\vec{x}(0)$. Equivalently, if A^t converges to the *zero matrix* as $t \to \infty$.
- **unstable** if $\vec{x}(t) \to \infty$ as $t \to \infty$ for some initial condition $\vec{x}(0)$. Equivalently, if some entry of A^t diverges as $t \to \infty$.
- **nonasymptotically stable** (we will *not* call these systems stable) if $\vec{x}(t)$ stays bounded (but does not necessarily go to 0) as $t \to \infty$, for *every* $\vec{x}(0)$.

  We will not distinguish between unstable and nonasymptotically stable. Both of these situations will be called 'not stable'.

4. In each of the following situations, we have a discrete dynamical system whose matrix A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$ and associated nonzero eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. Write down a formula for $\vec{x}(t)$ in terms of the given initial condition $\vec{x}(0)$, and then explain what happens to $\vec{x}(t)$ as $t \to \infty$.

(a) $\lambda_1 = 0.7, \lambda_2 = -0.3$, $\vec{x}(0) = \vec{v}_1 + \vec{v}_2$.

Solution: $\vec{x}(t) = (0.7)^t\vec{v}_1 + (-0.3)^t\vec{v}_2$. As $t \to \infty$, $\vec{x}(t) \to 0$. So this system is $\boxed{\textit{stable.}}$

(b) $\lambda_1 = 2, \lambda_2 = 1$, $\vec{x}(0) = v_1 - v_2$.

Solution: $\vec{x}(t) = 2^t\vec{v}_1 + 1^t\vec{v}_2$. As $t \to \infty$, $\vec{x}(t)$ grows without bound in magnitude. So this system is $\boxed{\textit{unstable.}}$

(c) $\lambda_1 = 1 + i, \lambda_2 = 1 - i, \lambda_3 = 1$, $\vec{x}(0) = v_1 + v_2 + v_3$.

Solution: $\vec{x}(t) = (1+i)^t\vec{v}_1 + (1-i)^t\vec{v}_2 + 1^t\vec{v}_2 = (\sqrt{2})^t e^{\pi i t/4}\vec{v}_1 + (\sqrt{2})^t e^{-\pi i t/4}\vec{v}_2 + \vec{v}_3$. As $t \to \infty$, this grows without bound in magnitude. Therefore, this system is $\boxed{\textit{unstable.}}$

(d) $\lambda_1 = 1, \lambda_3 = -1$, $\vec{x}(0) = 2v_1 + v_2$.

Solution: $\vec{x}(t) = 2(1)^t\vec{v}_1 + (-1)^t\vec{v}_2$. As t increases, this flips back and forth between $2\vec{v}_1 + \vec{v}_2$ and $2\vec{v}_1 - \vec{v}_2$. So this system is nonasymptotically stable - in particular, it is *not stable.*

5. Can you give a simple criterion to determine whether a matrix A is *asymptotically* stable?

   Solution: We have seen above that if A is diagonalizable, then A is asymptotically stable if and only if all of its eigenvalues λ satisfy $|\lambda| < 1$. We will see in the next class that this holds for nondiagonalizable matrices too. That is, A is asymptotically stable iff every eigenvalue λ satisfies $|\lambda| < 1$.

6. A mouse is in a maze with three rooms, with all three connected to each other. At time $t = 0$, the mouse is in room 1, and at each step, the mouse walks to one of the two adjacent rooms, chosen randomly. If $\vec{x}(t)$ is a vector containing the probabilities that the mouse is in each of the three rooms at time t, then this defines the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with $A = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ and $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (why?). Solve for $\vec{x}(t)$, and determine what happens as $t \to \infty$.[1]

   Solution: Calculate the characteristic polynomial:

$$\det \begin{bmatrix} -\lambda & 1/2 & 1/2 \\ 1/2 & -\lambda & 1/2 \\ 1/2 & 1/2 & -\lambda \end{bmatrix} = -\lambda^3 + \frac{3}{4}\lambda + 1/4 = -(\lambda - 1)\left(\lambda^2 + \lambda + \frac{1}{4}\right) = -(\lambda-1)(\lambda+1/2)^2$$

   so we have eigenvalues $1, -1/2, -1/2$. Calculating the eigenvectors,

$$\ker \begin{bmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\ker \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} = \operatorname{span} \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$$

   Now to calculate $\vec{x}(t)$, we can write $A = SDS^{-1}$ with $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, and exponentiate in the usual way, i.e. calculate

$$\vec{x}(t) = A^t \vec{x}(0) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^t & 0 \\ 0 & 0 & (1/2)^t \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

   Instead, I'm going to describe a second (equivalent) way of doing it. Write $\vec{x}(0)$ as a linear combination of the eigenvectors of A that we found.

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

[1] This is an example of a *random walk*, which is a special case of a *Markov process*, from probability. These discrete dynamical systems have applications all over the place. The Perron-Frobenius theorem gives conditions for such a system to be stable.

Now apply A^t: we know how A^t acts on each of these eigenvectors.

$$\begin{aligned}
A^t\vec{x}(0) &= \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + (-1/2)^t\frac{1}{3}\begin{bmatrix}1\\-1\\0\end{bmatrix} + (-1/2)^t\frac{1}{3}\begin{bmatrix}1\\0\\-1\end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix}1+2(-1/2)^t\\1-(-1/2)^t\\1-(-1/2)^t\end{bmatrix}
\end{aligned}$$

As $t \to \infty$, this converges to $\begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix}$. In other words, after the mouse has been wandering around in this maze for a long time, we should expect the mouse to be in each room with probability 1/3. This makes sense!

Note: We did this analysis with the example of a simple maze of three rooms with any two connected, and all transition probabilities equal to 1/2. But the same method works for ANY sort of random walk, or in general, any *Markov chain*. This, for example, is the theory underlying the Google Pagerank algorithm, among many, many other applications.