Math 21b, March 28: Complex Eigenvalues and Stability

1. Calculate the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Complex numbers: A complex number is a number of the form $a + bi$, where $i = \sqrt{-1}$. a is the *real part* and bi is the *imaginary part*. Addition of complex numbers is done componentwise, and multiplication is done using the fact that $i^2 = -1$.

$$(a + bi) + (c + di) = (a + c) + (b + d)i \qquad (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Euler's formula: Using Taylor series, one can show that $e^{i\theta} = \cos\theta + i\sin\theta$. This provides us a great tool to multiply complex numbers by using *polar coordinates*, i.e. expressing a complex number as $re^{i\theta} = r\cos\theta + ir\sin\theta$, where r is the *length* of the complex number. I.e., we use the formula

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

2. In this question, we'll use polar coordinates to calculate $(1 + i)^{10}$.

   (a) Write the complex number $1 + i$ in polar coordinates $r^{i\theta}$ by finding its length r and its angle θ.

   (b) Use this form to calculate $(1 + i)^{10}$. Can you find a geometric description of what multiplication by $1 + i$ does in the complex plane?

Complex conjugates: For any complex number $a + bi$, its *conjugate* is the complex number $a - bi$. In polar coordinates, the conjugate of $re^{i\theta}$ is $re^{-i\theta}$. Complex eigenvalues and eigenvectors of matrices come in *conjugate pairs.*

3. Let $A = \begin{bmatrix} -3 & -9 \\ 1 & -3 \end{bmatrix}$. In this problem, we will solve the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with initial condition $\vec{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Remember that we can derive $\vec{x}(t) = A^t\vec{x}(0)$, so we are essentially trying to calculate A^t.

   (a) Calculate the eigenvalues and eigenvectors of A. Use this to diagonalize A. (Your eigenvectors should have complex entries!)

   (b) Write down an expression for the matrix A^t in the form SDS^{-1} with D diagonal, and multiply this by $\vec{x}(0)$ to get $\vec{x}(t)$. (Note, the multiplication is quicker if you first multiply S^{-1} by $\vec{x}(0)$, instead of multiplying matrices.)

   (c) In words, how does $\vec{x}(t)$ behave as t increases? (Does it grow or shrink? Which quadrant will it be in?)

Stability: The matrix A associated to the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ is called

- **asymptotically stable** if $\vec{x}(t) \to 0$ as $t \to \infty$, for *every* initial condition $\vec{x}(0)$.
- **unstable** if $\vec{x}(t) \to \infty$ as $t \to \infty$ for some initial condition $\vec{x}(0)$.
- **nonasymptotically stable** if $\vec{x}(t)$ stays bounded (but does not necessarily go to 0) as $t \to \infty$, for *every* $\vec{x}(0)$.

4. In each of the following situations, we have a discrete dynamical system whose matrix A is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$ and associated nonzero eigenvectors $v_1, \dots, v_n$. Write down a formula for $\vec{x}(t)$ in terms of the given initial condition $\vec{x}(0)$, and then explain what happens to $\vec{x}(t)$ as $t \to \infty$.

   (a) $\lambda_1 = 0.7, \lambda_2 = -0.3$, $\vec{x}(0) = v_1 + v_2$.

   (b) $\lambda_1 = 2, \lambda_2 = 1$, $\vec{x}(0) = v_1 - v_2$.

   (c) $\lambda_1 = 1 + i, \lambda_2 = 1 - i, \lambda_3 = 1$, $\vec{x}(0) = v_1 + v_2 + v_3$.

   (d) $\lambda_1 = 1, \lambda_3 = -1$, $\vec{x}(0) = 2v_1 + v_2$.

5. Can you give a simple criterion to determine whether a matrix A is *asymptotically* stable?

6. A mouse is in a maze with three rooms, with all three connected to each other. At time $t = 0$, the mouse is in room 1, and at each step, the mouse walks to one of the two adjacent rooms, chosen randomly. If $\vec{x}(t)$ is a vector containing the probabilities that the mouse is in each of the three rooms at time t, then this defines the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with $A = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ and $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (why?). Solve for $\vec{x}(t)$, and determine what happens as $t \to \infty$.[1]

[1]This is an example of a *random walk*, which is a special case of a *Markov process*, from probability. These discrete dynamical systems have applications all over the place. The Perron-Frobenius theorem gives conditions for such a system to be stable.