Math 21b, March 23: Diagonalization

Diagonalizable matrices: An $n \times n$ matrix A is called *diagonalizable* if it is similar to a diagonal matrix, i.e. if there are $n \times n$ matrices S and D such that D is a diagonal matrix, and

$$A = SDS^{-1}$$

The entries $\lambda_1, \ldots, \lambda_n$ of D are the *eigenvalues* of A, and the columns $\vec{v}_1, \ldots, \vec{v}_n$ of S are the *eigenvectors* of A. Therefore, a matrix is diagonalizable if and only if it has an *eigenbasis*.

1. (Warmup) True or false:

   (a) If A is similar to B, then A^2 is similar to B^2.

   Solution: True. If $A = SBS^{-1}$, then $A^2 = (SBS^{-1})^2 = SBS^{-1}SBS^{-1} = SB^2S^{-1}$.

   (b) If A^2 is similar to B^2, then A is similar to B.

   Solution: False. For example, take $A = I_n$ and $B = -I_n$ (or take B to be any diagonal matrix whose diagonal entries are all ± 1).

   (c) If A is diagonalizable, then so is A^{100}.

   Solution: True. If $A = SDS^{-1}$, then $A^{100} = SD^{100}S^{-1}$, and the powers of a diagonal matrix are diagonal too (in fact, the product of two diagonal matrices is diagonal).

2. Suppose $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a basis of $\mathbb{R}^3$. Let A be a 3×3 matrix with eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ with respective eigenvalues $3, 2, -2$.

   (a) Find the $\mathfrak{B}$-coordinates of A.

   Solution: $A\vec{v}_1 = 3\vec{v}_1$, $A\vec{v}_2 = 2\vec{v}_2$, and $A\vec{v}_3 = -2\vec{v}_3$, so $[A]_{\mathfrak{B}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

   (b) How would you write a closed formula for A^{100}?

   Solution: $A^{100}\vec{v}_1 = 3^{100}\vec{v}_1$, $A^{100}\vec{v}_2 = 2^{100}\vec{v}_2$, and $A^{100}\vec{v}_3 = (-2)^{100}\vec{v}_3$, so $[A^{100}]_{\mathfrak{B}} = \begin{bmatrix} 3^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & (-2)^{100} \end{bmatrix}$.

   So $\boxed{A^{100} = S \begin{bmatrix} 3^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & (-2)^{100} \end{bmatrix} S^{-1}}$, where S is the matrix whose columns are $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

3. Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

   (a) Compute its eigenvalues, and give their algebraic multiplicity.

$$\det(A - \lambda I_3) = \det \begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (3-\lambda)^3$$

   Thus, A has only one eigenvalue $\lambda = 3$ with algebraic multiplicity 3.

   (b) Compute the eigenvectors. Is the matrix diagonalizable?

   $A - 3I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has kernel $\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rangle$, and so $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the only eigenvector (up to scaling), i.e. $\lambda = 3$ has *geometric multiplicity* equal to 1. So the matrix is NOT diagonalizable.

4. Consider the 2×2 rotation dilation matrix $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

   (a) Compute its eigenvalues.

   Solution:

$$\det(A - \lambda I_2) = \det \begin{bmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 + 16 = \lambda^2 - 6\lambda + 25 \implies \boxed{\lambda = 3 \pm 4i}$$

   (b) Are there eigenvectors? Why or why not?

   Solution: Thinking geometrically, there is no way that a vector in the plane can be scaled by a rotation around the origin! However, if we explicitly try to solve for the eigenvectors, i.e. compute the kernel of $A - (3+4i)I_2$ and $A - (3-4i)I_2$, we will find that there are 'eigenvectors' with entries that are *complex numbers*. We will address this point next week.

5. True or false:

   (a) If A and B have the same characteristic polynomial, then they are similar to each other. (Hint: think about problem 3 above.)

   Solution: False. For example, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, and $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ all have the same characteristic polynomial, but the first two cannot be similar because one is diagonalizable while the other isn't! (In fact, no two of the three matrices above are similar to one another.)

(b) If A and B are similar to each other, then $A + 2I_n$ and $B + 2I_n$ are similar to each other.

Solution: True. If $A = SBS^{-1}$, then

$$S(B + 2I_n)S^{-1} = SBS^{-1} + S(2I_n)S^{-1} = SBS^{-1} + 2SS^{-1} = A + 2I_n$$

(c) If A is a 3×3 matrix whose characteristic polynomial has roots $3, 3, 5$, then the matrix $A - 5I_2$ has rank two and the matrix $A - 3I_2$ has rank one.

Solution: False. Two possible to consider are $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. The first matrix satisfies the given conditions, but for the second matrix, $A - 3I_2$ has rank *two*, not one. (In general, for any 3×3 matrix A whose characteristic polynomial has roots $3, 3, 5$, $A - 5I_2$ will have rank two, but $A - 3I_2$ could have rank either one or two.)

6. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. (There is a trick to doing this without computing the characteristic polynomial.)

Solution: The sum of the entries of each row equals 6. Therefore, $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$, and we have found our first eigenvalue/eigenvector, i.e. $\boxed{\lambda_1 = 6, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$. Then notice that the first column plus the third column equals twice the second column, so $A \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and we have found our second eigenvalue/eigenvector, i.e. $\boxed{\lambda_2 = 0, \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$. Finally, see that $\mathrm{tr}(A) = 5$, and therefore because this equals the sum of the eigenvalues, the last eigenvalue is $\boxed{\lambda_3 = -1}$. To calculate the eigenvector associated to this eigenvalue, we calculate $\ker(A - (-1)I_3)$, i.e. the kernel of $\begin{bmatrix} 2 & 2 & 3 \\ 3 & 3 & 1 \\ 2 & 2 & 3 \end{bmatrix}$. We see that this kernel is spanned by the vector $\boxed{\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$.

7. In each part, decide whether there is a matrix with the given properties. If so, give an example; if not, explain why.

   (a) A diagonalizable 3×3 matrix which has 2 as an eigenvalue, trace 7, and determinant 12.

   Solution: Yes there is. Let the other two eigenvalues be λ_1, λ_2. Then $2 + \lambda_1 + \lambda_2 = 7 \implies \lambda_1 + \lambda_2 = 5$ and $2\lambda_1\lambda_2 = 12 \implies \lambda_1\lambda_2 = 6$. Thus, λ_2 and λ_3 are 2 and 3. The matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (or any matrix similar to it) satisfies the required properties.

   (b) A nondiagonalizable 3×3 matrix which has 2 as an eigenvalue, trace 7, and determinant 12.

   Solution: Yes there is. The matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (or any matrix similar to it) satisfies the required properties.

8. Let A be a noninvertible $n \times n$ matrix. Explain why 0 must be an eigenvalue, and find its geometric multiplicity in terms of $\text{rank}(A)$.

   Solution: 0 is an eigenvalue by definition, because $A - 0I_n = A$ has nontrivial kernel. The multiplicity is equal to the rank of the kernel, i.e. the *nullity*. By the rank-nullity theorem, this is equal to $n - \text{rank}(A)$.

9. Suppose that A is a 10×10 matrix and that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors of A with eigenvalues $5, 7, 13$, respectively.

   (a) Explain why $\vec{v}_2$ cannot be in $\text{span}(\vec{v}_1)$.

   Solution: If $\vec{v}_2$ were a multiple of $\vec{v}_1$, then it would be an eigenvector of eigenvalue 5, i.e. $A\vec{v}_2 = 5\vec{v}_2$. Since $A\vec{v}_2 = 7\vec{v}_2$, this is impossible, since $\vec{v}_2$ is nonzero.

   (b) Explain why $\vec{v}_3$ cannot be in $\text{span}(\vec{v}_1, \vec{v}_2)$.

   Solution: Suppose that $\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2$ for some c_1, c_2. Then we calculate $A\vec{v}_3$ in two ways.

$$A\vec{v}_3 = 13\vec{v}_3 = 13c_1\vec{v}_1 + 13c_2\vec{v}_2$$

$$A\vec{v}_3 = c_1A\vec{v}_1 + c_2A\vec{v}_2 = 5c_1\vec{v}_1 + 7c_2\vec{v}_2$$

   Therefore, these two expressions are equal. Taking their difference, we get $8c_1\vec{v}_1 + 6c_2\vec{v}_2 = 0$. But $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, so this is impossible unless $c_1 = c_2 = 0$. But that's also impossible, because $\vec{v}_3$ is nonzero.