Math 21b, March 23: Diagonalization

Diagonalizable matrices: An $n \times n$ matrix A is called *diagonalizable* if it is similar to a diagonal matrix, i.e. if there are $n \times n$ matrices S and D such that D is a diagonal matrix, and

$$A = SDS^{-1}$$

The entries $\lambda_1, \ldots, \lambda_n$ of D are the *eigenvalues* of A, and the columns $\vec{v}_1, \ldots, \vec{v}_n$ of S are the *eigenvectors* of A. Therefore, a matrix is diagonalizable if and only if it has an *eigenbasis*.

1. (Warmup) True or false:

   (a) If A is similar to B, then A^2 is similar to B^2.

   (b) If A^2 is similar to B^2, then A is similar to B.

   (c) If A is diagonalizable, then so is A^{100}.

2. Suppose $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a basis of $\mathbb{R}^3$. Let A be a 3×3 matrix with eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ with respective eigenvalues $3, 2, -2$.

   (a) Find the $\mathfrak{B}$-coordinates of A.

   (b) How would you write a closed formula for A^{100}?

3. Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

   (a) Compute its eigenvalues, and give their algebraic multiplicity.

   (b) Compute the eigenvectors. Is the matrix diagonalizable?

4. Consider the 2×2 rotation dilation matrix $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

   (a) Compute its eigenvalues.

   (b) Are there eigenvectors? Why or why not?

5. True or false:

   (a) If A and B have the same characteristic polynomial, then they are similar to each other. (Hint: think about problem 3 above.)

(b) If A and B are similar to each other, then $A + 2I$ and $B + 2I$ are similar to each other.

(c) If A is a 3×3 matrix whose characteristic polynomial has roots $3, 3, 5$, then the matrix $A - 5I_2$ has rank two and the matrix $A - 3I_2$ has rank one.

6. Find the eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. (There is a trick to doing this without computing the characteristic polynomial.)

7. In each part, decide whether there is a matrix with the given properties. If so, give an example; if not, explain why.

   (a) A diagonalizable 3×3 matrix which has 2 as an eigenvalue, trace 7, and determinant 12.

(b) A nondiagonalizable 3×3 matrix which has 2 as an eigenvalue, trace 7, and determinant 12.

8. Let A be a noninvertible $n \times n$ matrix. Explain why 0 must be an eigenvalue, and find its geometric multiplicity in terms of $\text{rank}(A)$.

9. In this question, you will think about why eigenvectors with different eigenvalues must be linearly independent. Suppose that A is a 10×10 matrix and that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors of A with eigenvalues $5, 7, 13$, respectively.

   (a) Explain why $\vec{v}_2$ cannot be in $\text{span}(\vec{v}_1)$.

   (b) Explain why $\vec{v}_3$ cannot be in $\text{span}(\vec{v}_1, \vec{v}_2)$.