MATH 21B, MARCH 9: DETERMINANTS - HOW TO COMPUTE THEM, AND WHAT THEY MEAN

Some properties of the determinant:

- $\det(A) = \det(A^T)$
- A is invertible $\iff \det(A) \neq 0 \iff$ A's rows (and columns) are linearly independent
- $\det(AB) = \det(A)\det(B)$. (see Question 7)
- $\det(I_n) = 1$, $\det(A^{-1}) = \frac{1}{\det(A)}$, $\det(A^k) = \det(A)^k$

Laplace Expansion: We can go along the entries in the first column of A and get an expression for the determinant

$$\det(A) = A_{11}\det(B_{11}) - A_{21}\det(B_{21}) + \ldots + (-1)^{n-1}A_{n1}\det(B_{n1})$$

where B_{i1} refers to the $(n-1)\times(n-1)$ matrix formed by removing the i-th row and first column of A. The same can be done with the entries in **any column** - if you do this starting with the r-th column instead, then you need to multiply by $(-1)^{r-1}$. Similarly, this can be done with **any row**.

(1) Compute the determinants using Laplace expansion.

(a) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}$

Solution: Expanding along the first row,

$$\det \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} = 1\det\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix} - 0\det\begin{bmatrix} 0 & 4 \\ 2 & -2 \end{bmatrix} + 3\det\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= (2-4) + 3(0+2) = \boxed{4}$$

(b) $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 7 & 1 & 0 \\ 1 & 2 & 4 & 1 \\ 5 & 10 & 2 & 5 \end{bmatrix}$

Solution: Expanding along the second row,

$$= -0\det\begin{bmatrix} 2 & 0 & -1 \\ 2 & 4 & 1 \\ 10 & 2 & 5 \end{bmatrix} + 7\det\begin{bmatrix} 1 & 0 & -1 \\ 1 & 4 & 1 \\ 5 & 2 & 5 \end{bmatrix} - 1\det\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \\ 5 & 10 & 5 \end{bmatrix} + 0\det\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 4 \\ 5 & 10 & 2 \end{bmatrix}$$

The first and fourth terms are zero. The third determinant is zero, because its second and third rows are multiples of one another (and therefore the rows are linearly dependent). Thus, our determinant equals

$$= 7\det\begin{bmatrix}1 & 0 & -1\\ 1 & 4 & 1\\ 5 & 2 & 5\end{bmatrix} = 7(1\cdot 4\cdot 5 - 1\cdot 1\cdot 2 - 1\cdot 1\cdot 2 + 1\cdot 4\cdot 5) = \boxed{252}$$

(2) True or false: the function $T : \mathbb{R}^4 \to \mathbb{R}$ defined by $T\left(\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}\right) = \det\begin{bmatrix}x_1 & x_2 & x_3 & x_4\\ 3 & -1 & 6 & 0\\ 2 & 3 & 0 & 5\\ -2 & 7 & 4 & 1\end{bmatrix}$ is linear.

Solution: True. If we Laplace expand along the first row, this determinant equals

$$= x_1\det\begin{bmatrix}-1 & 6 & 0\\ 3 & 0 & 5\\ 7 & 4 & 1\end{bmatrix} - x_2\det\begin{bmatrix}3 & 6 & 0\\ 2 & 0 & 5\\ -2 & 4 & 1\end{bmatrix} + x_3\det\begin{bmatrix}3 & -1 & 0\\ 2 & 3 & 5\\ -2 & 7 & 1\end{bmatrix} - x_4\det\begin{bmatrix}3 & -1 & 6\\ 2 & 3 & 0\\ -2 & 7 & 4\end{bmatrix}$$

$$= 212x_1 + 132x_2 - 84x_3 - 164x_4 = \begin{bmatrix}212 & 132 & -84 & 164\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

> **Some geometry:** The determinant of the $n \times n$ matrix $A = \begin{bmatrix}| & \cdots & |\\ \vec{v}_1 & \cdots & \vec{v}_n\\ | & \cdots & |\end{bmatrix}$ is equal to the *absolute value* of the n-dimensional volume of the **fundamental parallelepiped** in $\mathbb{R}^n$ whose edges are the vectors $\vec{v}_1, \ldots, \vec{v}_n$. In particular, if the columns of A are *linearly dependent*, then this parallelepiped is degenerate and $\det(A) = 0$.
>
> Since $\det(A) = \det(A^T)$, this is the same as the corresponding volume of the parallelepiped formed by the **row vectors** of A.

(3) Let A be an orthogonal matrix. What are the possible values for $\det(A)$?

Solution: The columns vectors of A are orthonormal, i.e. they are mutually perpendicular and all have length 1. Thus, the fundamental parallelepiped they form has volume 1. So $\boxed{\det(A) = \pm 1}$.

(4) Let $A = \begin{bmatrix}2 & 4 & 6\\ 1 & 2 & 1\\ 1 & 3 & 3\end{bmatrix}$.

(a) Row-reduce A, but keep track of each row operation that you used.

Solution: Using standard row-reduction techniques,

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \overset{\times(1/2)}{\to} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \overset{+}{\to} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \overset{s}{\to} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \overset{\times(-1/2)}{\to} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{+}{\to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I used $+$ to denote each time I added a multiple of one row to another, $\times$ each time I scaled a row by a constant, and s each time I swapped two rows.

(b) By interpreting each intermediate matrix as a parallelepiped generated by row vectors, how does the **volume** change with each step?

Solution: Intuitively,

- Scaling a row by a constant c will accordingly stretch (or shrink) the parallelepiped by a factor of c, and therefore will multiply the volume by c.
- Swapping two vectors doesn't change the parallelepiped, and therefore doesn't change the volume.
- Adding one row to another skews/shears the parallelepiped which does not change the volume (think in two dimensions: base times height).

(c) Correspondingly, can you calculate how the **determinant** changed at each step? Can you explain this using the **linearity** from problem 2?

Solution: If we calculate the determinants at every step above, we find they are

$$4 \overset{\times(1/2)}{\to} 2 \overset{+}{\to} 2 \overset{s}{\to} -2 \overset{\times(-1/2)}{\to} 1 \overset{+}{\to} 1$$

This suggests that scaling a row by c multiplies the determinant by c, swapping two rows multiplies the determinant by -1, and adding one row to another doesn't change the determinant.

The **firs**t of these properties (scaling a row) follows from the linearity property in problem 2. Since the determinant of an $n \times n$ matrix

$$T(\vec{x}) = \det \begin{bmatrix} - & \vec{x} & - \\ - & \vec{v}_2 & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix}$$

is a linear function of the first row $\vec{x}$ (thinking of $\vec{x}$ as a variable vector and the rest of the rows as fixed vectors), it follows by definition that by a scalar c will scale the determinant by c, i.e. $T(c\vec{x}) = cT(\vec{x})$. A similar argument holds when 'first row' is replaced by 'k-th row' for any k.

The **third** property (adding one row to another) also follows from the linearity property. For example, if we add the second row $\vec{v}_2$ to the first row, we get

$$T(\vec{x}+\vec{v}_2) = T(\vec{x}) + T(\vec{v}_2) = \det \begin{bmatrix} - & \vec{x} & - \\ - & \vec{v}_2 & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix} + \det \begin{bmatrix} - & \vec{v}_2 & - \\ - & \vec{v}_2 & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix}$$

The second determinant is zero because two rows are equal, and therefore, $T(\vec{x}+\vec{v}_2) = T(\vec{x})$, i.e. the determinant did not change.
The **second** property (swapping two rows) is not a consequence of the linearity from problem 2. However, we can prove it by simply expanding out the determinant by definition as a sum over all $n!$ permutation patterns, and notice that when we swap two rows in a permutation pattern, it either removes one upcrossing, or adds one new upcrossing - in either case, it flips the *sign* of the term corresponding to that permutation pattern. The consequence is that the determinant changes in sign.

Determinants and Row Reduction: Each row operation has a particular effect on the determinant.

- Multiplying a row or column by c, multiplies $\det(A)$ by c.
- Swapping two rows or two columns multiplies $\det(A)$ by -1.
- Adding a multiple of one row to another row (or of one column to another column) doesn't change $\det(A)$.

Therefore, during the row reduction of a square matrix A, if m swapping operations occurred and rows were scaled by factors $c_1, \ldots, c_k$, then

$$\det(A) = \frac{(-1)^m}{c_1 \cdots c_k} \det(\mathrm{rref}(A))$$

(5) Calculate the determinants using the easiest method you can. (Permutations, Laplace expansion, or Row reduction.)

(a) $\begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 2 & 0 & 4 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Solution: Laplace expand along the bottom row, then again along the bottom row of the new matrix.

$$= -1 \det \begin{bmatrix} 1 & 3 & 0 \\ 3 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \boxed{7}$$

(b) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$

Solution: Row reduce

$$\begin{bmatrix}1&1&1&1\\1&2&1&1\\1&1&3&1\\1&1&1&4\end{bmatrix} \xrightarrow{+} \begin{bmatrix}1&1&1&1\\0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix} \xrightarrow{\times(1/6)} \begin{bmatrix}1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \xrightarrow{+} I_4$$

so the original determinant is $\boxed{6}$.

(c) $\begin{bmatrix}0&1&0&0&0\\0&0&2&0&0\\3&0&0&0&0\\0&0&0&0&4\\0&0&0&5&0\end{bmatrix}^3$

Solution: There is only one permutation to worry about in the matrix shown (before being cubed), and it has three upcrossings (from $3-1$, $3-2$, $5-4$). Therefore, the matrix shown has determinant $(-1)^3 5! = -120$, and so after we cube this matrix, it has determinant $\boxed{-120^3 = -1728000}$.

(d) $\begin{bmatrix}0&4&3&1&0\\0&0&2&3&1\\0&2&4&3&0\\3&1&0&0&0\\0&1&0&0&0\end{bmatrix}$

Solution: Laplace expand along the bottom row, then the bottom row again, then the third column.

$$= -1\det\begin{bmatrix}0&3&1&0\\0&2&3&1\\0&4&3&0\\3&0&0&0\end{bmatrix} = 3\det\begin{bmatrix}3&1&0\\2&3&1\\4&3&0\end{bmatrix} = -3\det\begin{bmatrix}3&1\\4&3\end{bmatrix} = \boxed{-15}$$

(6) (a) Find the determinant of the upper triangular matrix $\begin{bmatrix}2&7&10&0&3\\0&-7&-10&-9&7\\0&0&8&3&-10\\0&0&0&-3&-5\\0&0&0&0&5\end{bmatrix}$.

Solution: We can Laplace expand along the bottom row repeatedly. Or, we may notice that the only permutation pattern which involves none of the zeroes below the diagonal, is the one going along the diagonal. Either way, we will get that the determinant is $\boxed{(2)(-7)(8)(-3)(5) = 1680}$.

(b) How would you find the determinant of any upper triangular matrix?

Solution: The determinant of any upper triangular matrix can be found the same way. Its determinant will be the **product of the diagonal entries.**

(c) What about the lower triangular matrix $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 7 & -7 & 0 & 0 & 0 \\ 10 & -10 & 8 & 0 & 0 \\ 0 & -9 & 3 & -3 & 0 \\ 3 & 7 & -10 & -5 & 5 \end{bmatrix}$?

Solution: Exactly the same.

(7) In this problem, we will prove that $\det(AB) = \det(A)\det(B)$.

(a) Consider the augmented matrix $[A|AB]$. Row-reduce until you have $[I_n|?]$ so that the matrix on the left side is I_n. What matrix do you have on the right side? (Hint: when you row-reduced $[A|b]$, you got $[I_n|A^{-1}b]$.)

Solution: Performing this row reduction gives us $A^{-1}AB$ on the right side, i.e. B. One way to see this is, if the columns of B are $\vec{v}_1, \ldots, \vec{v}_n$, then the columns of AB are $A\vec{v}_1, \ldots, A\vec{v}_n$, and so we are concurrently reducing each $[A|A\vec{v}_i] \to [I_n|\vec{v}_i]$. Another way to see it, is that each row operation corresponds to *multiplication on the left by a matrix.* As a concrete example, for 3×3 matrices,

- Scaling the third row of a matrix by c corresponds to $M \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix} M$.
- Swapping the second and third rows corresponds to $M \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} M$.
- Adding the second row to the first corresponds to $M \mapsto \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M$.

The sequence of row operations which takes A to I_n corresponds to multiplying on the left by A^{-1}. Therefore, this sequence of operations sends $AB \mapsto B$.

(b) How much did the operations you performed scale the determinant of the matrix on the left? How about the matrix on the right?

Solution: This process changed the determinant on the left from $\det(A)$ to 1, so it multiplied the determinant by $1/\det(A)$. We did the same row operations on both sides, so it did the same thing on the right. I.e., $\det(B) = \det(AB)/\det(A)$, which gives the product formula $\boxed{\det(AB) = \det(A)\det(B)}$.