MATH 21B, MARCH 9: DETERMINANTS - HOW TO COMPUTE THEM, AND WHAT THEY MEAN

Some properties of the determinant:

- $\det(A) = \det(A^T)$
- A is invertible $\iff \det(A) \neq 0 \iff A$'s rows (and columns) are linearly independent
- $\det(AB) = \det(A)\det(B)$. (see Question 7)
- $\det(I_n) = 1$, $\det(A^{-1}) = \frac{1}{\det(A)}$, $\det(A^k) = \det(A)^k$

Laplace Expansion: We can go along the entries in the first column of A and get an expression for the determinant

$$\det(A) = A_{11}\det(B_{11}) - A_{21}\det(B_{21}) + \ldots + (-1)^{n-1}A_{n1}\det(B_{n1})$$

where B_{i1} refers to the $(n-1)\times(n-1)$ matrix formed by removing the i-th row and first column of A. The same can be done with the entries in **any column** - if you do this starting with the r-th column instead, then you need to multiply by $(-1)^{r-1}$. Similarly, this can be done with **any row**.

(1) Compute the determinants using Laplace expansion.

(a) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 7 & 1 & 0 \\ 1 & 2 & 4 & 1 \\ 5 & 10 & 2 & 5 \end{bmatrix}$

(2) True or false: the function $T : \mathbb{R}^4 \to \mathbb{R}$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \det \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 3 & -1 & 6 & 0 \\ 2 & 3 & 0 & 5 \\ -2 & 7 & 4 & 1 \end{bmatrix}$ is linear.

Some geometry: The determinant of the $n \times n$ matrix $A = \begin{bmatrix} | & \cdots & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & \cdots & | \end{bmatrix}$ is equal to the *absolute value* of the n-dimensional volume of the **fundamental parallelepiped** in $\mathbb{R}^n$ whose edges are the vectors $\vec{v}_1, \dots, \vec{v}_n$. In particular, if the columns of A are *linearly dependent*, then this parallelepiped is degenerate and $\det(A) = 0$.

Since $\det(A) = \det(A^T)$, this is the same as the corresponding volume of the parallelepiped formed by the **row vectors** of A.

(3) Let A be an orthogonal matrix. What are the possible values for $\det(A)$?

(4) Let $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$.

(a) Row-reduce A, but keep track of each row operation that you used.

(b) By interpreting each intermediate matrix as a parallelepiped generated by row vectors, how does the **volume** change with each step?

(c) Correspondingly, can you calculate how the **determinant** changed at each step? Can you explain this using the **linearity** from problem 2?

Determinants and Row Reduction: Each row operation has a particular effect on the determinant.

- Multiplying a row or column by c, multiplies $\det(A)$ by c.
- Swapping two rows or two columns multiplies $\det(A)$ by -1.
- Adding a multiple of one row to another row (or of one column to another column) doesn't change $\det(A)$.

Therefore, during the row reduction of a square matrix A, if m swapping operations occurred and rows were scaled by factors $c_1, \ldots, c_k$, then

$$\det(A) = \frac{(-1)^m}{c_1 \cdots c_k} \det(\mathrm{rref}(A))$$

(5) Calculate the determinants using the easiest method you can. (Permutations, Laplace expansion, or Row reduction.)

(a) $\begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 2 & 0 & 4 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 5 & 0 \end{bmatrix}^3$

(b) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 4 & 3 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 3 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

(6) (a) Find the determinant of the upper triangular matrix $\begin{bmatrix} 2 & 7 & 10 & 0 & 3 \\ 0 & -7 & -10 & -9 & 7 \\ 0 & 0 & 8 & 3 & -10 \\ 0 & 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$.

(b) How would you find the determinant of any upper triangular matrix?

(c) What about the lower triangular matrix $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 7 & -7 & 0 & 0 & 0 \\ 10 & -10 & 8 & 0 & 0 \\ 0 & -9 & 3 & -3 & 0 \\ 3 & 7 & -10 & -5 & 5 \end{bmatrix}$?

(7) In this problem, we will prove that $\det(AB) = \det(A) = det(B)$.

(a) Consider the augmented matrix $[A|AB]$. Row-reduce until you have $[I_n|?]$ so that the matrix on the left side is I_n. What matrix do you have on the right side? (Hint: when you row-reduced $[A|b]$, you got $[I_n|A^{-1}b]$.)

(b) How much did the operations you performed scale the determinant of the matrix on the left? How about the matrix on the right?