MATH 21B, MARCH 2: GRAM-SCHMIDT, QR DECOMPOSITION, AND ORTHOGONAL MATRICES

Gram-Schmidt Orthonormalization: Given a set of vectors $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^m$, *Gram-Schmidt* is an algorithm to generate a set of *orthonormal* vectors $\vec{u}_1, \ldots, \vec{u}_n$ which have the *same span*. $\vec{u}_i$ is constructed from $\vec{v}_i$ recursively (starting with $\vec{u}_1$) by letting

$$\vec{w}_i = \vec{v}_i - \text{proj}_{V_{i-1}}(\vec{v}_i) \quad ; \quad \vec{u}_i = \vec{w}_i / ||\vec{w}_i||$$

where V_{i-1} is the space spanned by $\vec{u}_1, \ldots, \vec{u}_{i-1}$.

(1) Use Gram-Schmidt to orthonormalize the following sets of vectors.

(a) $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}.$

(b) $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}.$

QR decomposition: Let $m \geq n$. Any $m \times n$ matrix A whose columns are linearly independent can be written in the form $A = QR$

$$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & | & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}$$

where Q is an $m \times n$ matrix whose columns are *orthonormal* and R is an $n \times n$ matrix which is *upper triangular*. The vectors $\vec{u}_1, \ldots, \vec{u}_n$ are obtained from $\vec{v}_1, \ldots, \vec{v}_n$ by *Gram-Schmidt orthonormalization*.

(2) For each set of vectors in (1), write down the QR decomposition of $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$.

Transpose: Let A be an $m \times n$ matrix. Then the *transpose* A^T is an $n \times n$ matrix whose ij-th entry is $(A^T)_{ij} = A_{ji}$. The transpose satisfies the following properties:

- $(AB)^T = B^T A^T$.
- $\vec{v}^T \vec{w}$ is the dot product $\vec{v} \cdot \vec{w}$.
- $\vec{x} \cdot A\vec{y} = A^T\vec{x} \cdot \vec{y}$
- $(A^T)^T = A$
- If A is invertible, $(A^T)^{-1} = (A^{-1})^T$
- $\operatorname{rank}(A) = \operatorname{rank}(A^T)$

(3) Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 7 & 4 \\ 5 & 2 \end{bmatrix}$. Find a matrix B whose kernel is the orthogonal complement of $\operatorname{im}(A)$.

What can you say about $\operatorname{rank}(A)$ and $\operatorname{rank}(B)$?

Orthogonal Matrices: A matrix A is called *orthogonal* if it is square (say $n \times n$) and its columns form an orthonormal basis for $\mathbb{R}^n$. Equivalently, A is orthogonal if $A^T A = I_n$. (why?)

(4) Which of the following transformations are orthogonal? What geometric transformation does each represent? (One is rotation, one is reflection, one is projection, and one is shear.)

$$\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \qquad \frac{1}{3}\begin{bmatrix} 2 & -2 & -1 \\ -2 & -1 & -2 \\ -1 & -2 & 2 \end{bmatrix} \qquad \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

(5) Suppose A is an orthogonal $n \times n$ matrix, and $\vec{x}, \vec{y}$ are any vectors in $\mathbb{R}^n$. Argue that $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$. Can you give a geometric interpretation of this fact?