2013 WT1 Final Exam Breakdown

A: Multiple Choice

− A1: Finding inflection points
− A2: Inverse trig functions
✓ A3: Euler's Method
✓− A4: Newton's Method (Conceptual)
− A5: Linear Approximation
✓ A6: Slope Fields (Autonomous)
✓− A7: Periodic Functions
+ ~~A8~~: Qualitative Analysis (Autonomous)

B: Short Answer

B1: Implicit Differentiation ✓−
~~B2~~: Trig Related Rates ✓
B3: Minima & Maxima −
B4: Autonomous DE ✓+
B5: Limits / Derivative −
B6: Limits −
B7: Exponential Growth ✓−
B8: Limits −

C: Long Answer

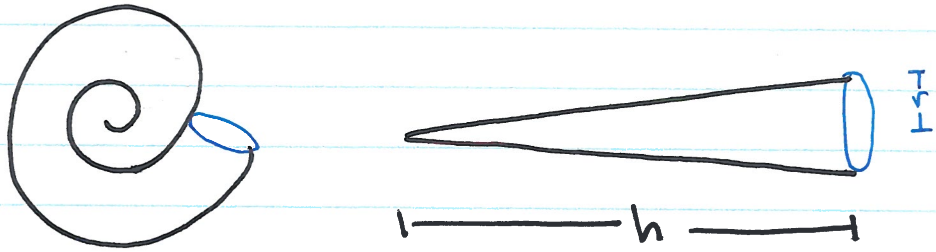
~~C1~~: Related Rates +

C2: Differential Equations (Autonomous) ✓

C3: Graph Sketching ✓−

~~C4~~: Optimization. ✓+ / +

C1. Mollusc



h grows at 0.1 cm/year

Q: At what rate will the volume grow when $h = 10$ cm and $r = 1$ cm

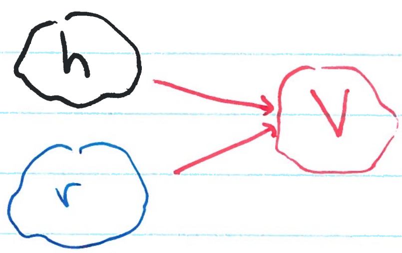
I am trying to find $\frac{dV}{dt}$.

$\frac{h}{r} = 10$

$h = 10r$

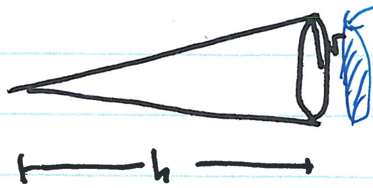
so $\frac{dh}{dt} = 10\frac{dr}{dt}$

Strategy:



• Take $\frac{d}{dt}$ of both sides of the equation.

Step 1:



Volume $= \frac{1}{3}h$ (Area of base)

$$V = \frac{1}{3}h(\pi r^2) = \frac{\pi}{3}(hr^2)$$

Step 2:

$$\frac{dV}{dt} = \frac{\pi}{3}\frac{d(hr^2)}{dt} = \frac{\pi}{3}\left(\frac{dh}{dt}r^2 + h\cdot 2r\frac{dr}{dt}\right) = \frac{\pi}{3}\left((0.1)\cdot 1^2 + 10\cdot 2\cdot 0.01\right)$$

$\frac{dh}{dt} = ?$	$\frac{dr}{dt} = ?$	$h = ?$	$r = ?$
0.1 (cm/year)	0.01 (cm/year)	10 cm	1 cm

$$= \frac{\pi}{3}(0.1 + 0.2)$$

$$= \boxed{\frac{\pi}{10}}$$

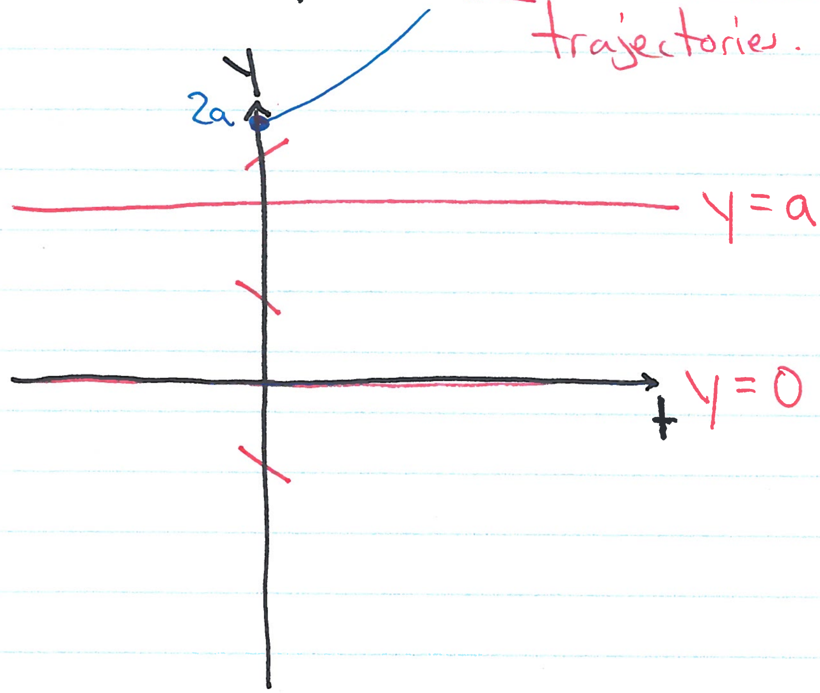
A8. Given $\frac{dy}{dt} = y^2(y-a)$ and $y(0) = 2a$

where $a > 0$ is a positive constant,

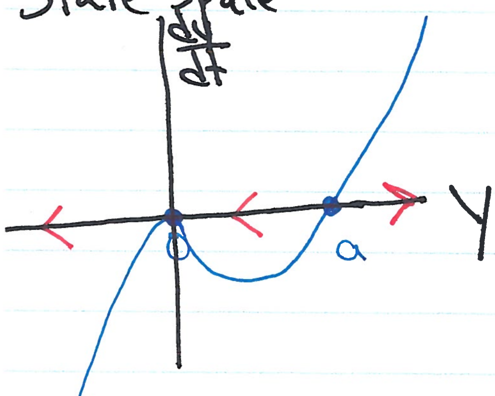
Goal: Find behavior of trajectories.

a) $\lim_{t\to\infty} y(t) =$

b) $= \lim_{t\to\infty} y(t) = \infty$



State Space



$\frac{dy}{dt} = y^2(y-a)$

Steady states: $y^2(y-a) = 0$

$\rightsquigarrow \boxed{y=0}, \boxed{y=a}$

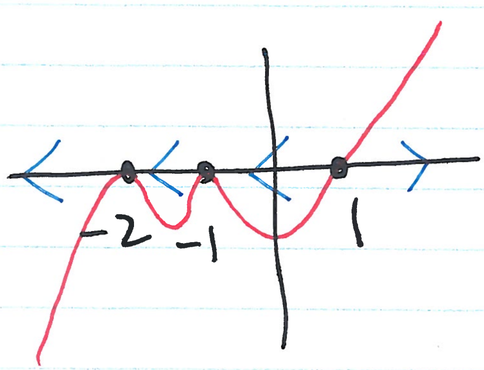
General: If you have $(y-1)^{\boxed{}}(y+1)^{\boxed{2}}(y+2)^{\boxed{4}}$,

odd even even

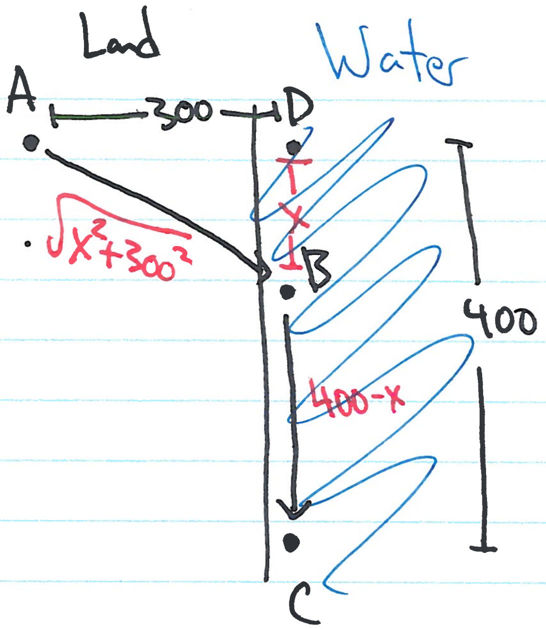
Roots 1 -1 -2

Even: doesn't cross axis

Odd: does cross



C4.



Energy to walk per meter = 2 (Energy to swim per meter)

Goal: Get from A to C minimal energy.

Let x = distance from D to B.
$(0 \le x \le 400)$

Goal:
- Express energy required in terms of x.
- Minimize that function (take derivative, set = 0, solve)

- Energy required = Energy walking + Energy swimming

$$2\sqrt{x^2+300^2} + 1(400-x)$$

$$\frac{d}{dx}\left(2(x^2+300^2)^{1/2} + 400 - x\right)$$

- Take derivative

$$2 \cdot \frac{1}{2}(x^2+300^2)^{-\frac{1}{2}} \cdot 2x - 1 = 0$$

$$\frac{300^2}{3} = \frac{300 \cdot 300}{3}$$

$$100 \cdot 300$$

$$\frac{2x}{\sqrt{x^2+300^2}} = 1$$

$$\sqrt{30{,}000}$$

$$\cancel{4}^{3}x^2 = \cancel{x^2} + \cancel{300^2}^{30{,}000} \Rightarrow \boxed{x = 100\sqrt{3}}$$

C2. $F(t)$ = # of fish in lake on day t.

$$\frac{dF}{dt} = \underbrace{I}_{} - \underbrace{\alpha N F}_{}$$

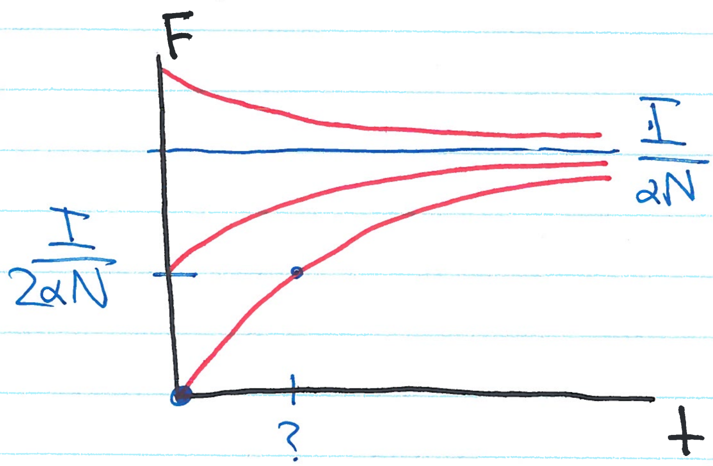
(I, α constants
N constant)

a) Company adds fish at constant rate I

Fishers catching fish at a rate proportional to F

b) What does α represent? Represents how effectively the fishers can catch fish.

c) Suppose N is constant. What is the steady state?



$I - \alpha N F = 0 \Rightarrow I = \alpha N F$

$$\boxed{\frac{I}{\alpha N} = F}$$

If you start with 0 fish, how long does it take to get to $\frac{1}{2}$(steady state?)

$$\boxed{F(t) = \frac{I}{\alpha N} + Ce^{-\alpha N t}}$$

$F(0) = 0 = \frac{I}{\alpha N} + C \rightsquigarrow C = -\frac{I}{\alpha N}$

$$\boxed{F(t) = \frac{I}{\alpha N} - \frac{I}{\alpha N} e^{-\alpha N t}}$$

$$F(t) = \frac{I}{2\alpha N} = \frac{I}{\alpha N} - \frac{I}{\alpha N} e^{-\alpha N t}$$

$$\frac{1}{2} = 1 - e^{-\alpha N t}$$

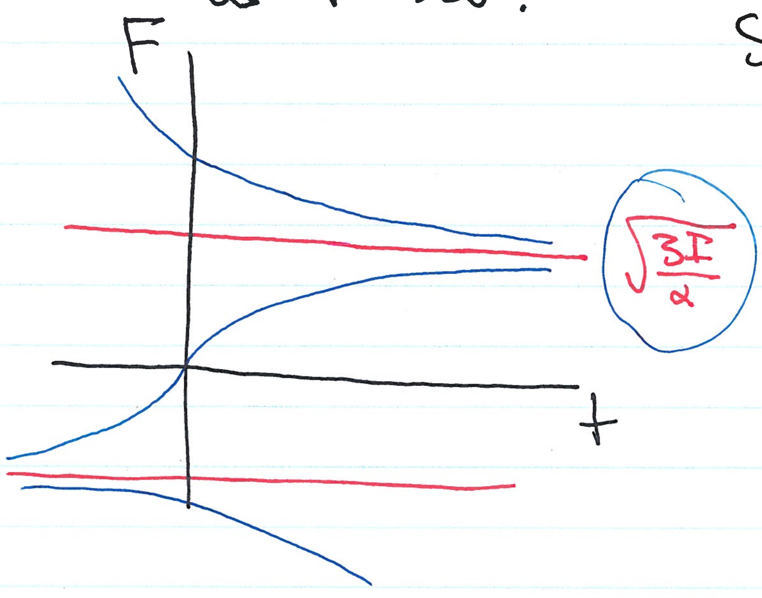
$$e^{-\alpha N t} = \frac{1}{2} \Rightarrow -\alpha N t = \ln\left(\frac{1}{2}\right) = -\ln(2)$$

$$\boxed{t = \frac{\ln(2)}{\alpha N}}$$

d) $\frac{dF}{dt} = I - \alpha N F$ (I, α constants $N = \frac{F}{3}$)

$$= I - \frac{\alpha F^2}{3}$$

Q: What is the steady state that F approaches as $t \longrightarrow \infty$?



Steady state: $I - \frac{\alpha F^2}{3} = 0$

$$I = \frac{\alpha F^2}{3}$$

$$\frac{3I}{\alpha} = F^2$$

$$\pm\sqrt{\frac{3I}{\alpha}} = F$$